SOME UNIQUE FIXED POINT THEOREMS

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In this paper some fixed point theorems have been proved which generalize the classical Banach's contraction mapping principle.

INTRODUCTION

A self-map f defined on a metric space (X, d) is called a contraction map if for some $0 < k < 1$,

$$d(f(x), f(y)) \leqslant kd(x, y), \text{ for all } x, y \in X.$$

Banach (1922) established the existence of a unique fixed point for a contraction map in a complete metric space. This celebrated principle has been generalized by many authors, viz., Chu and Diaz (1965), Sehgal (1969), Holmes (1969), Reich (1971), Hardy and Rogers (1973), Wong (1973) and others in various ways. In this paper we obtain yet another generalization of this principle.

MAIN RESULTS

Theorem 1—Let f be a continuous self-map defined on a complete metric space (X, d). Further, let f satisfy the following condition:

$$d(f(x), f(y)) \leqslant \frac{\alpha d(x, f(x)) \cdot d(y, f(y))}{d(x, y)} + \beta d(x, y) \qquad ..(A)$$

for all $x, y \in X$, $x \neq y$ and for some α, $\beta \in [0, 1)$ with $\alpha + \beta < 1$. Then f has a unique fixed point in X.

PROOF: Let x_0 be an arbitrary point of X and let $\{x_n\}$ where $x_n = f^n(x_0)$ and n is a positive integer, be the sequence of iterates of f at x_0. If $x_n = x_{n+1}$ for some n then the result is immediate. So let $x_n \neq x_{n+1}$ for all n. Now

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}))$$
$$\leqslant \frac{\alpha d(x_n, f(x_n)) \cdot d(x_{n-1}, f(x_{n-1}))}{d(x_n, x_{n-1})} + \beta\, d(x_n, x_{n-1})$$

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which implies that

$$d(x_{n+1}, x_n) \leqslant \left(\frac{\beta}{1-\alpha}\right) d(x_n, x_{n-1})$$

$$\vdots$$

$$\leqslant \left(\frac{\beta}{1-\alpha}\right)^n d(x_1, x_0).$$

By the triangle inequality we have for $m \geqslant n$,

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leqslant (k^n + k^{n+1} + \ldots + k^{m-1})\ d(x_0 f(x_0)), \quad \text{where } k = \frac{\beta}{1-\alpha} \\ &\leqslant \frac{k^n}{1-k} d(x_0, f(x_0)) \\ &\to 0 \quad \text{if} \quad m, n \to \infty. \end{aligned}$$

Since X is complete, there exists a $u \in X$ such that $x_n \to u$. Further, the continuity of f in X implies

$$\begin{aligned} f(u) &= f(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} f(x_n) \\ &= u. \end{aligned}$$

Therefore u is a fixed point of f in X. Now, if there exists another point $v \neq u$ in X such that $f(v) = v$, then

$$\begin{aligned} d(v, u) &= d(f(v), f(u)) \\ &\leqslant \frac{\alpha d(v, f(v)) \cdot d(u, f(u))}{d(v, u)} + \beta d(v, u) \\ &= \beta d(v, u) < d(v, u), \end{aligned}$$

a contradiction. Hence u is a unique fixed point of f in X.

Remark 1: It is not difficult to see that it suffices to assume that (A) holds for all $x, y \in M$, where M is a dense subset of X (Kannan 1969).

Remark 2: In Theorem 1 the condition that X is complete and f is continuous on X can be replaced by assuming that there exists an $x_0 (\in X)$ such that $\{f^n(x_0)\} \supset \{f^{n_i}(x_0)\} \to u \in X$ and f is continuous on the $cl\,\{f^n(x_0)\}$ (Edelstein 1962).

Example—Let $X = [0, 1]$ with the usual metric and let a map $f : X \to X$ be defined as follows:

$$f(x) = \begin{cases} x/4, & x \in [0, \frac{1}{2}) \\ x/5, & x \in [\frac{1}{2}, 1]. \end{cases}$$

It can be easily verified that for $\alpha = 3/5$ and $\beta = 19/50$, conditions of Remark 2 are satisfied with 0 as the only fixed point while f, being discontinuous, is not a contraction map.

In the next theorem, we establish the existence of a unique fixed point of a map f where we assume only the continuity of some iterate of f.

Theorem 2—Let f be a self-map defined on a complete metric space (X, d) such that (A) holds. If for some positive integer p, f^p is continuous, then f has a unique fixed point.

PROOF: Define a sequence $\{x_n\}$ as in Theorem 1. Clearly it converges to some point $u \in X$. Therefore its subsequence $\{x_{n_k}\}$, $(n_k = kp)$ also converges to u.

Also,

$$\begin{aligned} f^p(u) &= f^p(\lim_{k\to\infty} x_{n_k}) \\ &= \lim_{k\to\infty} x_{n_{k+1}} \\ &= u. \end{aligned}$$

Therefore u is a fixed point of f^p. We now show that $f(u) = u$. Let m be the smallest positive integer such that $f^m(u) = u$ but $f^q(u) \neq u$ $(q = 1, 2, \dots, m-1)$. If $m > 1$, then

$$\begin{aligned} d(f(u), u) &= d(f(u), f^m(u)) \\ &\leqslant \frac{\alpha d(u, f(u)) \cdot d(f^{m-1}(u), f^m(u))}{d(u, f^{m-1}(u))} + \beta d(u, f^{m-1}(u)), \end{aligned}$$

which implies that

$$d(f(u), u) \leqslant \frac{\beta}{1-\alpha} d(u, f^{m-1}(u)).$$

As

$$\begin{aligned} d(u, f^{m-1}(u)) &= d(f^m(u), f^{m-1}(u)) \\ &\leqslant \frac{\alpha d(f^{m-1}(u), f^m(u)) \cdot d(f^{m-2}(u), f^{m-1}(u))}{d(f^{m-1}(u), f^{m-2}(u))} \\ &\quad + \beta d(f^{m-1}(u), f^{m-2}(u)), \end{aligned}$$

it follows that

$$d\big(u, f^{m-1}(u)\big) = d\big(f^m(u), f^{m-1}(u)\big)$$

$$\leqslant \left(\frac{\beta}{1-\alpha}\right) d\big(f^{m-1}(u), f^{m-2}(u)\big)$$

$$\vdots$$

$$\leqslant \left(\frac{\beta}{1-\alpha}\right)^{m-1} d\big(f(u), u\big).$$

Therefore,

$$d\big(f(u), u\big) \leqslant \left(\frac{\beta}{1-\alpha}\right)^{m} d\big(f(u), u\big)$$

$$< d\big(f(u), u\big),$$

a contradiction. Hence $f(u) = u$.

The unicity of u as a fixed point of f follows as in Theorem 1.

Now we shall also relax condition (A) in Theorem 2. In fact the following theorem generalizes Theorem 1 as well.

Theorem 3—Let f be a self-map defined on a complete metric space (X, d) such that for some positive integer m, f satisfy the condition:

$$d\big(f^m(x), f^m(y)\big) \leqslant \frac{\alpha d\big(x, f^m(x)\big) \, . \, d\big(y, f^m(y)\big)}{d(x, y)} + \beta d(x, y) \qquad \ldots\text{(B)}$$

for all $x, y \in X$, $x \neq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. If f^m is continuous then f has a unique fixed point.

PROOF: That f^m has a unique fixed point u (say) in X follows from Theorem 1.

Also,

$$f(u) = f\big(f^m(u)\big)$$

$$= f^m\big(f(u)\big)$$

which implies that $f(u) = u$. Further, since a fixed point of f is also a fixed point of f^m and f^m has a unique fixed point u, it follows that u is a unique fixed point of f.

To show that Theorem 3 is more powerful than Theorem 1, we give the following example:

Example—Let $X = [0, 1]$ with the usual metric and let a map $f : X \to X$ be defined as follows:

$$f(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ \frac{1}{2}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

That f is discontinuous and does not satisfy (A) for any $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ when $x = \frac{1}{2}$, $y = 1$, can be easily seen. Now $f^2(x) = 0$ for all $x \in [0, 1]$. It can be verified that f^2 satisfy the conditions of Theorem 3 and 0 is a unique fixed point of f^2.

In the next theorem, we study the existence of a unique common fixed point of two mappings which are not necessarily continuous or commuting. Note that even if $X = [0, 1]$ and f_1, f_2 be two commuting continuous self-maps defined on X, f_1, f_2 need not have a common fixed point (refer to Smart 1974, Theorem 7.1.4).

Theorem 4—Let f_1 and f_2 be two self-maps defined on a complete metric space (X, d) satisfying the condition:

(i) For some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$,

$$d(f_1(x), f_2(y)) \leqslant \frac{\alpha d(x, f_1(x)) \,.\, d(y, f_2(y))}{d(x, y)} + \beta\, d(x, y)$$

for all $x, y \in X$, $x \neq y$.

(ii) $f_1 f_2$ is continuous on X.

(iii) there exists an $x_0 \in X$ such that in the sequence $\{x_n\}$, where

$$x_n = \begin{cases} f_1(x_{n-1}), & \text{when } n \text{ is even} \\ f_2(x_{n-1}), & \text{when } n \text{ is odd} \end{cases}$$

$x_n \neq x_{n+1}$ for all n.

Then f_1 and f_2 have a unique common fixed point.

PROOF: We have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(f_1(x_{2n-1}), f_2(x_{2n})) \\ &\leqslant \frac{\alpha d(x_{2n-1}, f_1(x_{2n-1})) \,.\, d(x_{2n}, f_2(x_{2n}))}{d(x_{2n-1}, x_{2n})} \\ &\quad + \beta d(x_{2n-1}, x_{2n}) \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leqslant \left(\frac{\beta}{1-\alpha}\right) d(x_{2n-1}, x_{2n}) \\ &\;\;\vdots \\ &\leqslant \left(\frac{\beta}{1-\alpha}\right)^{2n} d(x_0, x_1). \end{aligned}$$

Similarly we can show that

$$d(x_{2n+1}, x_{2n+2}) \leqslant \left(\frac{\beta}{1-\alpha}\right)^{2n+1} d(x_0, x_1).$$

Now it can be easily seen that $\{x_n\}$ is a cauchy sequence. Let $\{x_n\} \to u$, then the subsequence $\{x_{n_k}\} \to u$, where $n_k = 2k$.

$$\begin{aligned} \text{Now } f_1 f_2(u) &= f_1 f_2 (\lim_{k\to\infty} x_{n_k}) \\ &= \lim_{k\to\infty} x_{n_{k+1}} \\ &= u. \end{aligned}$$

We now show that $f_2(u) = u$. If $f_2(u) \neq u$, then

$$\begin{aligned} d(f_2(u), u) &= d(f_2(u), f_1 f_2(u)) \\ &\leqslant \frac{\alpha\, d(u, f_2(u)) \,.\, d(f_2(u), f_1 f_2(u))}{d(u, f_2(u))} + \beta\, d(u, f_2(u)) \\ &= (\alpha + \beta)\, d(u, f_2(u)), \end{aligned}$$

a contradiction, since $\alpha + \beta < 1$. Hence $f_2(u) = u$. Also

$$\begin{aligned} d(f_1(u), u) &= d(f_1(f_2(u)), u) \\ &= 0, \end{aligned}$$

which shows that $f_1(u) = u$.

If possible, let $v (\neq u) \in X$ be such that $f_1(v) = v$. Then,

$$\begin{aligned} d(v, u) &= d(f_1(v), f_2(u)) \\ &\leqslant \frac{\alpha d(v, f_1(v)) \,.\, d(u, f_2(u))}{d(v, u)} + \beta\, d(v, u) \\ &= \beta d(v, u) < d(v, u), \end{aligned}$$

a contradiction. Hence u is a unique fixed point of f_1. It is easy to check that u is also a unique fixed point of f_2.

Remark 3: If $f_1 = f_2 = f$, then Theorem 4 becomes a particular case of Theorem 2.

Lastly, we study the stability of fixed points for a sequence of functions. This problem was principally motivated by the work of Nadler (1968).

Theorem 5—Let f_n be a self-map defined on a metric space (X, d) with u_n as a fixed point for each $n = 1, 2, \ldots\ldots$. If f_n satisfies (A) for each n and $\{f_n\}$ converges pointwise to f, then $u_n \to u$ if and only if u is a fixed point of f.

PROOF: If $u = u_n$ for some n, the assertions follow easily. Hence $u \neq u_n$ for any n. Let $u_n \to u$, then

$$\begin{aligned} d(u, f(u)) &\leqslant d(u, u_n) + d(u_n, f_n(u)) + d(f_n(u), f(u)) \\ &= d(u, u_n) + d(f_n(u_n), f_n(u)) + d(f_n(u), f(u)) \\ &\leqslant d(u, u_n) + \frac{d(u_n, f_n(u_n)) \,.\, d(u, f_n(u))}{d(u_n, u)} + \beta d(u_n, u) \\ &\quad + d(f_n(u), f(u)) \\ &= (1+\beta)\, d(u_n, u) + d(f_n(u), f(u)) \ \to 0 \text{ as } n \to \infty; \end{aligned}$$

and hence $f(u) = u$.

Conversely, we assume that $f(u) = u$, then

$$\begin{aligned} d(u_n, u) &= d(f_n(u_n), f(u)) \\ &\leqslant d(f_n(u_n), f_n(u)) + d(f_n(u), f(u)) \\ &\leqslant \frac{\alpha\, d(u_n, f_n(u_n)) \,.\, d(u, f_n(u))}{d(u_n, u)} + \beta d(u_n, u) \\ &\quad + d(f_n(u), f(u)) \end{aligned}$$

which implies that

$$\begin{aligned} d(u_n, u) &\leqslant \frac{1}{1-\beta} d(f_n(u), f(u)) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $u_n \to u$. This completes the proof of the theorem.

Note: In fact, it is easy to verify that f and f_n ($n = 1, 2, \ldots$) can have only unique fixed points.

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