ALGEBRAIC CLASSIFICATION OF SPACE-MATTER TENSOR IN GENERAL RELATIVITY*

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In the present paper, the author gives an algebraic classification of the space-matter tensor, in general relativity, which was first introduced by Petrov (1969). A comparison of the present classification with that of Petrov has been made and it is found that the cases III (*a*) and III (*b*) correspond to the case of gravitational radiation. As an example, the author deals with the space found inside an incompressible sphere. In the appendix, the forms of the curvature tensor (using Weyl tensor) for the different cases has been given.

1. INTRODUCTION

We assume that the metric

$$ds^2 = g_{ab}\, dx^a\, dx^b$$

of the space-time V_4 is reducible at a point to the Galilian form

$$ds^2 = -(dx_1)^2 - (dx_2)^2 - (dx_3)^2 + (dx_4)^2.$$

Let the Einstein's field equations be

$$R_{ab} - \tfrac{1}{2} R g_{ab} = \lambda T_{ab} \tag{1}$$

where λ is a constant and T_{ab} is the energy-momentum tensor. On contraction (1) yields

$$\lambda T = -R \tag{2}$$

Introduce a fourth order tensor (Petrov 1969)

$$A_{abcd} = \lambda/2\ \ (g_{ac}\, T_{bd} + g_{bd}\, T_{ac} - g_{ad}\, T_{bc} - g_{bc} T_{ad}). \tag{3}$$

From the definition this tensor has the following properties:

$$A_{abcd} = -A_{bacd} = -A_{abdc} = A_{cdab} \tag{4}$$

$$A_{abcd} + A_{acbd} + A_{adbc} = 0. \tag{5}$$

Contraction of (3) over b and d yields

$$A_{ac} = \lambda T_{ac} + \lambda/2 T g_{ac} = \lambda T_{ac} - R/2 g_{ac} \tag{6}$$

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Define a new fourth order tensor [2]

$$P_{abcd} = R_{abcd} - A_{abcd} + \sigma (g_{ac} g_{bd} - g_{ad} g_{bc}) \tag{7}$$

This tensor is known as space-matter tensor. The first part of this tensor represents the curvature of the space and the second part represents the distribution and motion of the matter. This tensor has the following properties:

(i) $P_{abcd} = - P_{bacd} = - P_{abdc} = P_{cdab}$

$P_{abcd} + P_{acdb} + P_{adbc} = 0$

(ii) $P_{ac} = R_{ac} - \lambda T_{ac} + R/2\, g_{ac} + 3\sigma g_{ac}$

$= (R + 3\sigma)\, g_{ac}$

(iii) If the distribution and motion of the matter, *i.e.*, T_{ab} and the space-matter tensor P_{abcd} are given, then R_{abcd}, the curvature of the space is determined to within the scaler σ.

(iv) If $T_{ab} = 0$ and $\sigma = 0$, then P_{abcd} is the curvature of the empty space-time.

(v) If g_{ab}, the metric tensor, σ, the scaler and P_{abcd} are known, then T_{ab} can be determined uniquely.

Following Géhéniau and Debever (1956) the Riemann curvature tensor may be decomposed in the following form

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd} \tag{8}$$

where C_{abcd} is the Weyl tensor, E_{abcd} is the Einstein curvature tensor defined by

$$E_{abcd} = - \tfrac{1}{2} (g_{ac} S_{bd} + g_{bd} S_{ac} - g_{ad} S_{bc} - g_{bc} S_{ad}) \tag{9}$$

where

$$S_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R \tag{10}$$

being the trace-less Ricci tensor, and G_{abcd} is defined by

$$G_{abcd} = - R/12 \, (g_{ac} g_{bd} - g_{ad} g_{bc}) \tag{11}$$

From eqns. (9), (10) and (11), (8) can be written as

$$R_{abcd} = C_{abcd} + \tfrac{1}{2} (g_{ad} R_{bc} + g_{bc} R_{ad} - g_{ac} R_{bd} - g_{bd} R_{ac}) - R/6 \, (g_{ad} g_{bc} - g_{ac} g_{bd}) \tag{12}$$

and from eqns. (1), (3) may be expressed as

$$A_{abcd} = \tfrac{1}{2} (g_{ac} R_{bd} + g_{bd} R_{ac} - g_{ad} R_{bc} - g_{bc} R_{ad}) - R/2 \, (g_{ac} g_{bd} - g_{ad} g_{bc}) \tag{13}$$

From eqns. (12) and (13), (7) may be expressed as

$$P_{abcd} = C_{abcd} + (g_{ad} R_{bc} + g_{bc} R_{ad} - g_{ac} R_{bd} - g_{bd} R_{ac}) + (2/3\, R + \sigma)(g_{ac} g_{bd} - g_{ad} g_{bc}) \tag{14}$$

2. Classification of the Space-Matter Tensor P_{abcd}

We, now give a scheme for the classification of the space-matter tensor P_{abcd}. If, in the decomposition (14), we take Ricci tensor R_{ab} and the scaler σ equal to zero, then the space-matter tensor P_{abcd} reduces to Weyl tensor C_{abcd}. Thus the classification of the space-matter tensor P_{abcd} is equivalent to the classification of the Weyl tensor C_{abcd} in empty space-time.

Depending upon the number of independent eigenvalues of a complex six dimensional symmetric tensor, we arrive at the following classification (Sharma and Husain 1969) of the Weyl tensor.

Case I—When all the three eigenvalues are different, then the matrix form for C_{abcd} reduces to

$$C_{AB} = \begin{bmatrix} U & V \\ V & -U \end{bmatrix}, \qquad A, B, = 1, 2, \ldots, 6$$

where U and V are three by three matrices given by

$$U = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}; \qquad V = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}$$

Case II—When two of the three eigenvalues are equal, then we have the following two cases:

Case II (a)—The matrices U and V are given by

$$U = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & -2\alpha_1 \end{bmatrix}; \qquad V = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & -2\beta_1 \end{bmatrix}$$

Case II (b)—The matrices U and V are as follows:

$$U = \begin{bmatrix} 2\alpha & 0 & 0 \\ 0 & -(\alpha+\delta) & -\gamma \\ 0 & -\gamma & -(\alpha-\delta) \end{bmatrix}; \qquad V = \begin{bmatrix} -2\beta & 0 & 0 \\ 0 & (\beta-\gamma) & -\delta \\ 0 & -\delta & (\beta+\gamma) \end{bmatrix}$$

Case III—When all the three eigenvalues are equal, then we have the following two cases:

Case III (a)—The matrices U and V are given by

$$U = \begin{bmatrix} 0 & -\alpha & -\beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{bmatrix}; \quad V = \begin{bmatrix} 0 & -\beta & \alpha \\ -\beta & 0 & 0 \\ \alpha & 0 & 0 \end{bmatrix}$$

Case III (b)—The matrices U and V are given by

$$U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & -\beta \\ 0 & -\beta & -\alpha \end{bmatrix}; \quad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta & \alpha \\ 0 & \alpha & -\beta \end{bmatrix}$$

Comparing our classification with that of Petrov (1969) we find that cases I, II (b) and III (a) correspond to the types I, II and III respectively. Moreover, case III (a) and III (b) belong to the case of gravitational radiation (Sharma and Husain 1969).

As an example, we consider the space found inside an incompressible sphere of radius r_0 and mass M with the metric

$$ds^2 = -dx_1^2 + dx_2^2 - dx_3^2 - (x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2/a^2 - r^2$$

$$+ (3h - h_0/2hh_0)^2\, dx_4^2$$

where $a = r_0\ (r_0/2m)^{\frac{1}{2}}$, $m = k\ M/c^2$, $1/h^2 = 1 - (2mr^2/r_0^3)$,

$$1/h_0^2 = 1 - (2m/r_0).$$

The energy-momentum tensor is given by

$$T_{ab} = (u_0 + p/c^2)\, u_a\, u_b - pg_{ab},$$

where u_0 is the rest-mass-density and p is the pressure.

Here, after calculations, we notice that all the three eigenvalues are equal and therefore the space found inside an incompressible sphere of radius r_0 and mass M belongs to the Petrov type III or case III (a) of the present classification.

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APPENDIX

Using Weyl tensor

$$C_{abcd} = R_{abcd} - \lambda/2\,(g_{ac}T_{bd} - g_{ad}T_{bc} + g_{bd}T_{ac} - g_{bc}T_{ad}) + \lambda/3\,T\,(g_{ac}g_{bd} - g_{ad}g_{bc})$$

with cases I, II (*a*), II (*b*), III (*a*) and III (*b*), we derive the following expressions for the curvature tensor of the following five possible cases. Now

$$R_{AB} = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}, \quad A, B = 1, 2, \ldots, 6$$

where X, Y, Y^* and Z are three by three non-singular matrices with Y^* as the transpose of Y.

Case I

$$X = \begin{bmatrix} \alpha_1 + \frac{1}{2}(T_{11} - T_{44}) + 1/3T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & \alpha_2 + \frac{1}{2}(T_{22} - T_{44}) + 1/3T & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & \alpha_3 + \frac{1}{2}(T_{33} - T_{44}) + 1/3T \end{bmatrix}$$

$$Y = \begin{bmatrix} \beta_1 & \frac{1}{2}T_{34} & -\frac{1}{2}T_{24} \\ -\frac{1}{2}T_{34} & \beta_2 & \frac{1}{2}T_{14} \\ \frac{1}{2}T_{24} & -\frac{1}{2}T_{14} & \beta_3 \end{bmatrix}$$

$$Z = \begin{bmatrix} -\alpha_1 - \frac{1}{2}(T_{22} + T_{33}) - 1/3T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & -\alpha_2 - \frac{1}{2}(T_{11} + T_{33}) - 1/3T & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & -\alpha_3 - \frac{1}{2}(T_{11} + T_{22}) - 1/3T \end{bmatrix}$$

Y^* = Transpose of Y.

Case II (a)

$$X = \begin{bmatrix} \alpha_1 + \frac{1}{2}(T_{11} - T_{44}) + 1/3T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & \alpha_1 + \frac{1}{2}(T_{22} - T_{44}) + 1/3T & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & -2\alpha_1 + \frac{1}{2}(T_{33} - T_{44}) + 1/3T \end{bmatrix}$$

$$Y = \begin{bmatrix} \beta_1 & \frac{1}{2}T_{34} & -\frac{1}{2}T_{24} \\ -\frac{1}{2}T_{34} & \beta_1 & \frac{1}{2}T_{14} \\ \frac{1}{2}T_{24} & -\frac{1}{2}T_{14} & -2\beta_1 \end{bmatrix}$$

$$Z = \begin{bmatrix} -\alpha_1 - \frac{1}{2}(T_{22}+T_{33}) - 1/3T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & -\alpha_1 - \frac{1}{2}(T_{11}+T_{33}) - 1/3T & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & 2\alpha_1 + \frac{1}{2}(T_{11}+T_{22}) + 1/3T \end{bmatrix}$$

$Y^* =$ Transpose of Y.

Case II (b)

$$X = \begin{bmatrix} 2\alpha + \frac{1}{2}(T_{11}-T_{44}) + 1/3T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & -(\alpha+D) + \frac{1}{2}(T_{22}-T_{44}) + 1/3T & \frac{1}{2}T_{23} + E \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} + E & -(\alpha-D) + \frac{1}{2}(T_{22}-T_{44}) + 1/3T \end{bmatrix}$$

$$Y = \begin{bmatrix} -2\beta & \frac{1}{2}T_{34} & -\frac{1}{2}T_{24} \\ -\frac{1}{2}T_{34} & \beta + E & \frac{1}{2}T_{14} + D \\ \frac{1}{2}T_{24} & -\frac{1}{2}T_{14} + D & \beta - E \end{bmatrix}$$

$$Z = \begin{bmatrix} -2\alpha - \frac{1}{2}(T_{22}+T_{33}) - 1/3T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & (\alpha-D) - \frac{1}{2}(T_{11}+T_{33}) - 1/3T & \frac{1}{2}T_{23} + E \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} - E & (\alpha+D) - \frac{1}{2}(T_{11}+T_{22}) - 1/3T \end{bmatrix}$$

$Y^* =$ Transpose of Y and $D = -\delta$, $E = -\gamma$.

Case III (a)

$$X = \begin{bmatrix} \frac{1}{2}T_{11} & -\alpha + \frac{1}{2}(T_{11}-T_{12}) + 1/3T & -\beta + \frac{1}{2}(T_{11}-T_{13}) + 1/3T \\ -\alpha + \frac{1}{2}(T_{11}-T_{12}) + 1/3T & \frac{1}{2}T_{22} & \frac{1}{2}T_{23} \\ -\beta + \frac{1}{2}(T_{11}-T_{13}) + 1/3T & \frac{1}{2}T_{23} & \frac{1}{2}T_{33} \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -\beta & -\tfrac{1}{2}T_{24} + \alpha \\ -\beta & 0 & \tfrac{1}{2}T_{14} \\ \tfrac{1}{2}T_{24} + \alpha & -\tfrac{1}{2}T_{14} & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} -\tfrac{1}{2}T_{11} & \alpha - \tfrac{1}{2}(T_{11} + T_{12}) - 1/3T & \beta - \tfrac{1}{2}(T_{11} + T_{13}) - 1/3T \\ \alpha - \tfrac{1}{2}(T_{11} + T_{12}) - 1/3T & -\tfrac{1}{2}T_{22} & -\tfrac{1}{2}T_{23} \\ \beta - \tfrac{1}{2}(T_{11} + T_{13}) - 1/3T & -\tfrac{1}{2}T_{23} & -\tfrac{1}{2}T_{33} \end{bmatrix}$$

Y^* = Transpose of Y.

Case III (b)

$$X = \begin{bmatrix} \tfrac{1}{2}T_{11} & \tfrac{1}{2}T_{12} & \tfrac{1}{2}T_{13} \\ \tfrac{1}{2}T_{12} & \alpha + \tfrac{1}{2}(T_{22} - T_{44}) + 1/3T & -\beta + \tfrac{1}{2}(T_{22} - T_{23}) + 1/3T \\ \tfrac{1}{2}T_{13} & -\beta + \tfrac{1}{2}(T_{22} - T_{23}) + 1/3T & -\alpha + \tfrac{1}{2}(T_{33} - T_{44}) + 1/3T \end{bmatrix}$$

$$Y = \begin{bmatrix} \tfrac{1}{2}T_{14} & \tfrac{1}{2}T_{34} & -\tfrac{1}{2}T_{24} \\ -\tfrac{1}{2}T_{34} & \beta & \tfrac{1}{2}T_{14} + \alpha \\ \tfrac{1}{2}T_{24} & -\tfrac{1}{2}T_{14} + \alpha & -\beta \end{bmatrix}$$

$$Z = \begin{bmatrix} -\tfrac{1}{2}T_{11} & -\tfrac{1}{2}T_{12} & -\tfrac{1}{2}T_{13} \\ -\tfrac{1}{2}T_{12} & -\alpha - \tfrac{1}{2}(T_{11} + T_{33}) - 1/3T & \beta - \tfrac{1}{2}(T_{11} + T_{23}) - 1/3T \\ -\tfrac{1}{2}T_{13} & \beta - \tfrac{1}{2}(T_{11} + T_{23}) - 1/3T & \alpha + \tfrac{1}{2}(T_{11} + T_{22}) - 1/3T \end{bmatrix}$$

Y^* = Transpose of Y.

The above five cases show how the space-time curvature depends on the functions defining the physical state.