COLLINEATIONS IN FINITE PROJECTIVE PLANES AND GROUP DIVISIBLE DESIGNS

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Constructions of some PBIB designs of group divisible type and constructions of some partially balanced orthogonal matrices have been given using central collineations in finite projective planes. The converse result is established in two cases. As an application, it is shown that it is possible to obtain non-isomorphic solutions of the m-BOMs and PBIBDs. Lastly, using the known necessary conditions for the existence of designs, some necessary conditions for the existence of a central collineation of a particular order for a given projective plane are obtained.

1. INTRODUCTION

Most of the notations in this paper can be found in Raghavarao (1971). Bhaskararao (1970) has defined a balanced orthogonal matrix (BOM). This definition is extended to an m-balanced orthogonal matrix (m-BOM) by Hebbare and Patwardhan (1975).

Definition 1.1—A $(\pm 1, 0)$, matrix is said to be an m-balanced orthogonal matrix with m associate classes (m-BOM) if, (i) any two distinct rows of A are orthogonal. (ii) A^*A is an incidence matrix of a PBIBD with m associate class, where $A^*B = (a_{ij}b_{ij})$ is the Schur product of $A = (a_{ij})$ and $B = (b_{ij})$ of the same order.

We shall require only 2-BOMs in which the corresponding design is an affine resolvable semiregular group divisible design (2-BOM-ARSRGDD) or a regular group divisible design (2-BOM-RGDD).

2. THE MAIN RESULTS

Theorem 2.1—Given that a projective plane (PP) π of order n exists and given a translation of π of orders, there exists an ARSRGDD with the following parameters:

$$\left.\begin{aligned} &v = b = sm^2,\ r = k = sm, \\ &m' = sm,\ n' = m,\ \lambda_1 = 0,\ \lambda_2 = s \\ &t = sm,\ \beta = m \text{ where } m = \frac{n}{s}. \end{aligned}\right\} \qquad \ldots(2.1)$$

v treatments are divided into m' rows of n' treatments each and b blocks are divided into t classes of β blocks each. If s is even, then there also exists a 2-BOM-ARSRGDD with the parameters (2.1).

PROOF: Let c be the translation of π of order s with centre P on an axis M. There are n lines labelled K_i ($i = 1, \ldots, n$) through P other than M. Similarly, there are n points labelled Q_i ($i = 1, \ldots, n$) on M other than P. The line K_i is fixed by c and the points on K_i are permuted in transitive sets. $c, c^2 \ldots, c^{s-1}$ are all (P, M)-translations none of which is the identity collineation. It, therefore, follows that, for a point R on K_i, $R \neq P$, $R, Rc, \ldots, Rc^{s-1}$ are all distinct points on K_i. Every transitive set has s points and hence $s|n$. Let $m = n/s$. Thus we have m transitive sets on K_i numbered from 1 to m. Let a fixed representative point of the jth transitive set on the line K_i be denoted by P_{ij}, $i = 1, \ldots, n = ms$; $j = 1, \ldots, m$. This labelling is then dualized on the lines through Q_i. The lines through Q_i are permuted in transitive sets of s lines each and hence we get L_{kl} as a representative line of the lth transitive set through Q_k; $k = 1, \ldots, n = sm$; $l = 1, \ldots, m$.

Let G be the cyclic group of order s generated by c. Let s be even, say $s = 2q$.

Define incidence numbers by

$$\begin{aligned} a_{ij,kl} &= +1 \text{ if } P_{ij} \in L_{kl}c^u \text{ for some even number } u, \\ & \quad u = 0, 2, \ldots, 2q-2, \\ &= -1 \text{ if } P_{ij} \in L_{kl}c^u \text{ for some odd number } u,\ u = 1, 3, \ldots, 2q-1, \\ &= 0 \text{ otherwise.} \end{aligned} \quad \ldots(2.2)$$

We prove the following lemmas,

Lemma 2.1 — $\sum_l (a_{ij,kl})^2 = 1$

and

$$\sum_{k,l} (a_{ij,kl})^2 = n = sm.$$

PROOF: This follows from the fact that the point P_{ij} is on exactly one line through Q_k.

Lemma 2.2 — $\sum_j (a_{ij,kl})^2 = 1$,

and

$$\sum_{i,j} (a_{ij,kl})^2 = n = sm.$$

PROOF: Dual of Lemma 2.1.

Lemma 2.3 — $\sum_{k,l} a_{ij,\,kl}\, a_{ij',\,kl} = 0$ for $j \neq j'$.

PROOF: A non-zero summand corresponds to $P_{ij}\, x \in L_{kl}$ and $P_{ij'}\, y \in L_{kl}$ for some $x, y \in G$, But this is impossible since $P_{ij}\, x$ and $P_{ij'}\, y$ already lie on K_i and hence cannot lie on any other line in π.

Lemma 2.4 — $\sum_{i,j} a_{ij,\,kl}\, a_{ij,\,kl'} = 0$ for $l \neq l'$.

PROOF: Dual of Lemma 2.3.

Lemma 2.5 — $\sum_{k,l} a_{ij,\,kl}\, a_{i'j',\,kl} = 0$ for $i \neq i'$.

PROOF: It is obvious that for a non-zero summand, we need consider only those incidences of lines passing through P_{ij} and $P_{i'j'}\, x$ since lines of the same class pass through $P_{ij}\, y$ and $P_{i'j'}\, xy \;\forall\; y \in G$.

Let the line passing through P_{ij} and $P_{i'j'}$ be $L_{kl}\, x$. Then either both $a_{ij,\,kl}$ and $a_{i'j',\,kl}$ are $+1$ or both are -1. It follows that $a_{ij,\,kl}\, a_{i'j',\,kl} = +1$ for this line. Let the line $L_{k'l'}\, y$ pass through P_{ij} and $P_{i'j'}\, c$, then $a_{ij,\,k'l'}\, a_{i'j',\,k'l'} = -1$. Thus the summands for $P_{ij}\, P_{i'j'}$ and $P_{ij}\, P_{i'j'}\, c$ cancel each other. This holds for all the summand pairs $P_{ij}\, P_{i'j'}\, c^u$ and $P_{ij}\, P_{i'j'}\, c^{u+1}$ for $u = 0, 2, \ldots, 2q-2$. Hence the Proof.

Lemma 2.6 — $\sum_{i,j} a_{ij,\,kl}\, a_{ij,\,k'l'} = 0$ for $k \neq k'$.

PROOF: Dual of Lemma 2.5.

If we write,

$$A = (a_{ij,\,kl})$$

where i and j correspond to rows of A and k and l corresponds to columns of A. Then A is orthogonal with

$$AA^T = sm\, I_{sm^2}. \qquad \ldots(2.3)$$

Let $A^*A = B = (b_{ij,\,kl})$;

then $b_{ij,\,kl} = +1$ if $P_{ij} \in L_{kl}\, x$ for some $x \in G$, $\qquad \ldots(2.4)$

$= 0$ otherwise.

B can be directly defined using the incidence relation (2.4) and without using the Schur product. This definition is valid for odd s as well.

It can be seen that the new incidence numbers also satisfy Lemmas 2.1, 2.2, 2.3 and 2.4. Lemmas 2.5 and 2.6 change to

Lemma 2.7 — $\sum_{k,l} b_{ij,kl} b_{i',j',k,l} = s$ if $i \neq i'$.

Lemma 2.8 — $\sum_{i,j} b_{ij,kl} b_{ij,k'l'} = s$ if $k \neq k'$.

Above lemmas imply that B is an incidence matrix of the ARSRGDD with parameters (2.1).

The proof of the next theorem is similar.

Theorem 2.2—Given that a PP π of order n exists and given a homology of π of order s, there exists an RGDD with the parameters,

$$v = b = m(sm + 2), r = k = sm + 1$$

$$m' = sm + 2, \ n' = m, \ \lambda_1 = 0 \quad \lambda_2 = s \qquad \text{...(2.5)}$$

where

$$m = \frac{n-1}{s}.$$

If s is even, there also exists a 2-BOM-RGDD with the parameters (2.5).

Equivalence of the existence of an SRGDD with parameters (2.1) to the existence of an orthogonal array $A = (sm^2, sm, m, 2)$ has been shown by Bose *et al.* (1953). Hence Theorem 2.2 can be used to construct orthogonal arrays. However, limitations of these constructions are evident in the light of the fact that all the known finite projective planes have prime power orders.

Some applications of the main results are presented in the next three sections.

3. Non-Isomorphic Solutions of Some PBIBDs and 2-BOMs

A pertinent inquiry is the investigation of the converse of Theorems 2.1 and 2.2. In general, it is impossible to trace the steps of their proofs backwards because of the inadequacy of the incidence relations. However, it is possible in the cases $s = 1$ and 2 as per the following theorems.

Theorem 3.1—The existence of a finite PP of order n is equivalent to the existence of an ARSRGDD with the following parameters:

$$v = b = n^2, \ r = k = n$$

$$m' = n, \ n' = n, \ \lambda_1 = 0, \ \lambda_2 = 1. \qquad \text{...(3.1)}$$

Theorem 3.2—The existence of a finite PP of order n is equivalent to the existence of an RGDD with the following parameters:

$$v = b = n^2 - 1, \ r = k = n$$

$$m' = n + 1, \ n' = n - 1, \ \lambda_1 = 0, \ \lambda_2 = 1. \qquad \text{...(3.2)}$$

The proofs of both the theorems are trivial.

Theorem 3.3—The existence of a translational involution of a PP of order n is equivalent to the existence of a 2-BOM-ARSRGDD with the parameters,

$$v = b = 2m^2, \; r = k = 2m$$

$$m' = 2m, \; n' = m., \quad \lambda_1 = 0, \; \lambda_2 = 2 \qquad ...(3.3)$$

where

$$m = \frac{n}{2} \, .$$

PROOF: One part of the proof is contained in Theorem 2.1. The converse is the following.

We are given a $(\pm 1, 0)$ matrix A of order $2m^2$ with $n = 2m$. Also,

$$AA^T = 2mI_{2m^2}.$$

Let $B = A^*A$, then B is a matrix of an ARSRGDD with parameters as in (3.3). In short, the matrices A and B satisfy the corresponding lemmas in section 2.

Define a PP π as follows. M is a line in π and P is a point on M. Q_i, $i = 1, \ldots, n$ are other points on M. K_i, $i = 1, \ldots, n$ are other lines through P. Points on K_i are P_{ij}, $P_{ij}\,c$, $i = 1, \ldots, 2m$; $j = 1, \ldots, m$. Similarly lines through Q_i are L_{ij}, $L_{ij}\,c$, $i = 1, \ldots, 2m$; $j = 1, \ldots, m$. Thus we have $(n^2 + n + 1)$ lines in π. Define the incidence relation by

$$P_{ij} \in L_{kl} \Leftrightarrow P_{ij}\,c \in L_{kl}\,c \Leftrightarrow a_{ij,\,kl} = +1$$

$$P_{ij} \in L_{kl}\,c \Leftrightarrow P_{ij}\,c \in L_{kl} \Leftrightarrow a_{ij,\,kl} = -1. \qquad ...(3.4)$$

Then Lemma 2.2 shows that every line L_{kl} or $L_{kl}c$ contains n points of π other than Q_i. Hence it follows that every line in π contains exactly $(n + 1)$ points. The Lemmas 2.1, 2.3, 2.5 and 2.7 are sufficient to show that any two distinct points determine exactly one line passing through them. Hence,

(i) There are exactly $(n^2 + n + 1)$ lines in π.

(ii) There are exactly $(n + 1)$ points on every line in π.

(iii) Any two points in π uniquely determine the line passing through them.

Therefore π is a PP of order n with c as translation.

The proof of the next theorem is similar.

Theorem 3.4—The existence of a homology involution of a PP of order n is equivalent to the existence of a 2-BOM-RGDD with the parameters,

$$v = b = 2m(m + 1), \; r = k = 2m + 1$$

$$m' = 2(m + 1), n' = m, \; \lambda_1 = 0, \; \lambda_2 = 2 \qquad ...(3.5)$$

where

$$m = \frac{n-1}{2}.$$

It is obvious that the designs obtained from the non-isomorphic PPs should be non-isomorphic and *vice versa* by above theorems. Accordingly,

Corollary 3.1—There is a one-to-one correspondence between the non-isomorphic PPs of order n and the non-isomorphic ARSRGDDs with the parameters (3.1).

Corollary 3.2—There is a one-to-one correspondence between the non-isomorphic PPs of order n and the non-isomorphic RGDDs with the parameters (3.2).

Corollary 3.3—There is a one-to-one correspondence between the non-isomorphic PPs of order n having a translational involution and the non-isomorphic 2-BOM-ARSRGDDs with the parameters (3.3).

Corollary 3.4.—There is a one-to-one correspondence between the non-isomorphic PPs of order n having a homology involution and the non-isomorphic 2-BOM-RGDDs with the parameters (3.5).

4. Applications of Corollary 3.3

(i) Take $n = 16$. The desarguesian plane, say π_1 of order 16, exists. Also, Hall plane (Hall 1943), say π_2 of order 16, exists. Both π_1 and π_2 are translation planes and are non-isomorphic. In both the planes, there exists a point-line flag $P \in M$ such that the plane is $P{-}M$ transitive. $G(P, M)$ is transitive on the points of a line through P and hence $|G(P, M)|$ is divisible by 16. Therefore, order of $G(P, M)$ is even and thus $G(P, M)$ must have an involution. It, then, follows that we get two non-isomorphic 2-BOM-ARSRGDDs with the parameters,

$$v = b = 128, \ r = k = 16$$
$$m' = 16, \ n' = 8, \ \lambda_1 = 0, \ \lambda_2 = 2. \qquad \ldots(4.1)$$

(ii) Consider 2-BOM-ARSRGDD with the parameters,

$$v = b = 8, \ r = k = 4$$
$$m' = 4, \ n' = 2, \ \lambda_1 = 0, \ \lambda_2 = 2. \qquad \ldots(4.2)$$

This is obtainable from a PP of order 4 which is unique. Therefore, a 2-BOM-ARSRGDD with the parameters an in (4.2) exists and is unique.

(iii) The uniqueness of a PP of order 8 implies the uniqueness of 2-BOM-ARSRGDD with the parameters,

$$v = b = 32, \ r = k = 8$$
$$m' = 8, \ n' = 4, \ \lambda_1 = 0, \ \lambda_2 = 2. \qquad \ldots(4.3)$$

5. Non-existence of Certain Central Collineations

The necessary conditions for the existence of ARSRGDD and RGDD in terms of Hilbert norm residue symbol (a, b) have been obtained by Shrikhande and Raghavarao (1963) and Bose and Connor (1952) respectively. These conditions, when applied to the designs obtained in Theorems 2.1 and 2.2 respectively, give the following corollaries.

Corollary 5.1—A necessary condition for a PP of order n to have a translation of order s is that

(i) $n = sm$ for some positive integer m.

(ii) If (i) is satisfied, then

$$(-1, m)_p^{sm} (-1, sm)_p^K (s, m)_p^{sm} = +1$$

for all odd primes p,

where $K = sm(m-1)(sm^2 - sm + 3)/2$.

Corollary 5.2—A necessary condition for a PP of order n to have a homology of order s is that

(i) $n - 1 = sm$ for some positive integer m.

(ii) If (i) is satisfied, then

$$(-1, sm+1)_p^K (m, sm+1)_p^{sm} = +1$$

for all odd primes p, where

$$K = (sm+2)(m-1)[(sm+2)(m-1)+1]/2.$$

For a PP of order n to exist, it must satisfy the Bruck-Ryser (1949) conditions. As can be seen trivially, for $s = 1$, Corollaries 5.1 and 5.2 give the Bruck-Ryser conditions. Excluding the cases in which the Bruck-Ryser conditions are not satisfied and those in which the above conditions are identically satisfied, we get the following corollaries:

Corollary 5.3—Let n be a positive integer for which the Bruck-Ryser conditions are satisfied then necessary conditions for a PP of order n (if it exists) to have a translation of odd orders are that

(5.3.1) $n = sm$ for some positive integer m.

(5.3.2) If (5.3.1) is satisfied, and

(*a*) if $m \equiv 1 \pmod 4$, then

$$(-s, n)_p = +1 \text{ for all odd primes } p.$$

(*b*) if $m \equiv 3 \pmod 4$, then

$(s, -n)_p = +1$ for all odd primes p.

Corollary 5.4—Let n be a positive integer for which the Bruck-Ryser conditions are satisfied, then necessary conditions for a PP of order n (if it exists) to have a homology of odd order s are that

(5.4.1) $n - 1 = sm$ for some positive integer m.

(5.4.2) If (5.4.1) is satisfied, and

(*a*) if $m \equiv 1 \pmod 4$, then

$(m, n)_p = +1$ for all odd primes p.

(*b*) if $m \equiv 3 \pmod 4$, then

$(-m, n)_p = +1$ for all odd primes p.

Examples—The following central collineations are non-existent:

Sl. No.	Order of the plane — n	Order of the collineation	Type of the central collineation	Reason
1.	24	5	translation	5.3.1
2.	15	3	translation	5.3.2 (*a*)
3.	45	3	translation	5.3.2 (*b*)
4.	28	5	homology	5.4.1
5.	40	3	homology	5.4.2 (*a*)
6.	1240	21	homology	5.4.2 (*b*)

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