BINORMAL OPERATORS II

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Let (BN) denote the class of all bounded linear operators on a Hilbert space H such that T^*T and TT^* commute. It is proved that if $T \in (BN)$ and $T^k = T$, for some positive integer $k \geqslant 2$, then T^{k-1} is a projection. It is also shown how it is possible that $T^m \in (BN)$ for $m \in I$, I is an arbitrary index set.

This paper is a continuation of an earlier paper (Arun Bala 1977). In that paper we studied bounded linear operators T acting on a Hilbert space H such that T^*T and TT^* commute. Such operators are called binormal and the class of such operators is denoted by (BN). In the present paper we sharpen the theorem proved earlier (Arun Bala 1976). Let $R(T)$ and $N(T)$ denote the range and null space of T respectively.

We proved the following theorem (Arun Bala 1977):

Theorem A—If $T \in (BN)$ and T is idempotent then T is self-adjoint.

Now we show that we can weaken the idempotency in the above Theorem A.

Theorem 1—If $T \in (BN)$ and satisfies the condition

$$T^k = T$$

some positive integer $k \geqslant 2$ then T^{k-1} is a projection.

To prove the above theorem we need the following Lemma:

Lemma—Suppose $T^k = T$ and $T \in (BN)$, then $T^{k-1} \in (BN)$ and $T^{k-2} \in (BN)$.

PROOF: Since $T^k = T$ therefore T is one-one on $R(T)$. $T \in (BN)$ this implies that orthogonal projections on $R(T)$, $N(T)$ commute. Thus $N(T) \perp R(T)$ and $T = T_1 \oplus O$, where T_1 is one-one and with dense range. Then $T_1^k = T_1 \Rightarrow T_1^{k-1} = I \Rightarrow T_1^{k-1} = I \oplus O \in (BN)$. Thus $R(T_1)$ is closed and T_1 is invertible. Now $T \in (BN) \Rightarrow T_1 \in (BN) \Rightarrow T_1^{-1} \in (BN)$. But $T_1^{k-1} = I \Rightarrow T_1^{k-2} = T_1^{-1}$. This implies that $T_1^{k-2} \in (BN)$. Thus $T^k = T$, $T \in (BN)$, imply $T^{k-1} \in (BN)$ and T^{k-} $\in (BN)$.

PROOF OF THE THEOREM 1: $T^{2(k-1)} = T^{k-2}\,T = T^{k-1}$ so that T^{k-1} is an idempotent operator. Also by the above Lemma $T^{k-1} \in (BN)$. Therefore T^{k-1} is a projection by Theorem A.

Theorem 2—Let A be such that $A^m \in (BN)$ for $n \in I$, I is a set of non-negative integers.

Define

$$T_{n,A} = \begin{bmatrix} 0 & 0 & & & \\ A & 0 & & & \\ 0 & A & & & \\ & & \ddots & & \\ & & & A & 0 \end{bmatrix}$$

where $T_{n,A}$ is a $n \times n$ block matrix acting on $\sum_{i=1}^{n} \oplus H_i$, $H_i = H$, $A \in B(H)$. [$B(H)$ is the algebra of all bounded operators on H].

Then $T^m_{n,A} \in (BN)$, for $m \in I\,U\left[\frac{n}{2}, \infty\right)$.

PROOF:

$$T^m_{n,A} = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} m\text{-zeros} & & & \\ A^m & & & \\ & A^m & & \underbrace{0 \cdots 0}_{m\text{-zeros}} \end{bmatrix}$$

Thus

$$T^m_{n,A}\; T^{*m}_{n,A} = \left[\begin{array}{c|c} \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} & m \times m \\ \hline & \begin{matrix} A^m A^{*m} & & \\ & \ddots & \\ & & A^m A^{*m} \end{matrix} \end{array}\right]$$

$$T^{*m}_{n,A}\; T^m_{n,A} = \left[\begin{array}{c|c} \begin{matrix} A^{*m} A^m & & \\ & \ddots & \\ & & A^{*m} A^m \end{matrix} & \\ \hline & \left.\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}\right\} m \times m \end{array}\right]$$

clearly $T^{m}_{n,A} \in (BN)$ iff $A^{m} \in (BN)$ for $m \geqslant n/2$. Since $T_{n,A}\, n = 0$ we actually have a stronger counter example in that

$$T \in (BN),\ T^{m} \text{ normal for } m \geqslant n_0 \not\Rightarrow T^{k} \in (BN) \text{ for all } k \leqslant n_0.$$

Now take a fixed $n_0 \geqslant 2$. There exists $A_{n_0} \in (BN)$ such that $A_{n_0}^{m} \in (BN)$ iff $m = 1, \ldots\ldots n_0 - 1$.

Then $T^{m}_{2n_0} + 2,\ A_{n0} \in (BN)\ \forall\ m \neq n_0$.

Let $T_{n0} = T_{2n+2},\ A_{n0}$.

Then for any set of positive integers I let

$$T_I = \sum_{n0 \in I} \oplus T_{n0}.$$

Then $T_I \in (BN)$, $T_I^{m} \in (BN)$ for precisely those $m \in I$. Note $\Sigma \oplus$ is an orthogonal sum.

Furuta and Nakamoto (1971) have proved that if $T^k = T$, $k > 2$ and T is either normaloid or spectraloid then T is normal and partial isometry. We show that this is not true in the case of binormal operators.

For example let $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

Then $T^k = T$ for $k = 3$ and $T \in (BN)$. It is easy to see that T is neither normal nor is it partial-isometry.

We know that if a partial-isometry is paranormal then it turns out to be subnormal. We show that if a partial-isometry is binormal then T is not necessarily subnormal. Let $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then T is partial-isometry and $T \in (BN)$ but T is not subnormal (Halmos 1967).

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