ON GROUP STRUCTURE

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In this note the author tries to characterise the alternating group of order 12.

The aim of this paper is to give negative answer to the following problem posed in my previous paper (Kumar 1976).

Problem 1.1—Let H be a nilpotent group of class at most two such that $(3, |H|) = 1$ and $(3, |\text{Aut } H|) \neq 1$. Does there exist a Schreier's extension G of H such that each element of $G - H$ is of order three?

Finally, we have given a characterization of alternating group A_4 of order 12.

PRELIMINARIES

Here is preliminaries needed for the discussion (Lal 1976).

Let H be a subgroup of a group G and S a set of right coset representative system of G modulo H containing identity. (Call such a system an admissible coset representative system). Suppose $x, y \in S$ and $h \in H$. Then $x.y = f(x, y)\, x\, o\, y$ (. is the operation of G) and $xh = \sigma_x(h)\, x\, \theta h$. This gives an action θ of S on H, a map $S^2 \xrightarrow{f} H$, an operation o on S, a map $S \xrightarrow{\sigma} H^H$ denoted by $\sigma(x) = \sigma_x$. If H is normal, $G/H \simeq S$.

Call an extension G of H normal cubic if $G/H \simeq Z_3$. Now an interesting theorem of Kumar (1976 *a*) is as given below:

Theorem 1.2—If G is a normal cubic extension of H such that $(|H|, 3) = 1$ and for $|H|$ admissible coset representative systems, f, is trivial. Then H is nilpotent group of class at most two.

It has been proved, under the hypothesis of the theorem, that each element of $G - H$ is of order three. Thus if H is a group such that three does not divide its order and which has a Schreier's extension G of degree three, then H is a nilpotent group of class at most two. Then Aut H contains an element of order three. Thus converse problem is to see the Problem 1.1. Here we show that this is not true in general, for the existence of such an extension amounts to the existence of a $f \in$ Aut H, of order three satisfying $hf(h)f^2(h) = 1$. Kumar (1976) has proved this under some restrictions on H. (When H is a prime cyclic group of order $3k + 1$ or when H contains fixed-point-free automorphism of order three).

Example 1.3—Let $G = \{\pm 1, \pm i, \pm j \pm k\}$ be the Hamiltonian group of order eight in which operation is defined by the following table.

	1	−1	i	j	k	$-i$	$-j$	$-k$
1	1	−1	i	j	k	$-i$	$-j$	$-k$
−1	−1	1	$-i$	$-j$	$-k$	i	j	k
i	i	$-i$	−1	k	$-j$	1	$-k$	j
j	j	$-j$	$-k$	−1	i	k	1	$-i$
k	k	$-k$	j	$-i$	−1	$-j$	i	1
$-i$	$-i$	i	1	$-k$	j	−1	k	$-j$
$-j$	$-j$	j	k	1	$-i$	$-k$	−1	i
$-k$	$-k$	k	j	i	1	j	$-i$	−1

It is obvious that its commutator subgroup is Z_2. Hence G is a nilpotent group of class two. It is a well-established fact that Aut $G \simeq S_4$. Aut G contains an element of order three. Obviously -1 is the unique element of order two. Thus $f(-1) = -1$ for any $f \in$ Aut G, *i.e.*, Aut G contains no fixed-point-free automorphism. This group satisfies the conditions of Problem 1.1, *i.e.* $(3, |H|) = 1$ and $(3, |\text{Aut } G|) \neq 1$. Choose an element $f \in$ Aut G of order three. Then for $-1 \in G$, $(-1)f(-1)f^2(-1) = -1 \neq 1$. Hence there does not exist any Schreier's extension G' of G of degree three such that each element of $G' - G$ is of order three.

Example 1.4—Let G be the cyclic group of order $2n$ such that $(n, 3) = 1$ and $(3, |\text{Aut } G|) \neq 1$. (Such type of group exist for Z_{14} is such an example.) Then represent $Z_{2n} = \{\bar{o}, \bar{1}, \ldots, \bar{n}, \ldots, \overline{2n-1}\}$. Then $\bar{n}$ is the only element of order two. Hence for any $f \in$ Aut G, $f(\bar{n}) = \bar{n}$. Hence G has no Schreier's extension of degree three such that each element in the complementary set is of order three.

In view of this we have,

Proposition 1.5—Let G be the dihedral group D_{8n} of order $8n$. Then Aut G contains no fixed-point-free automorphism. $(n \geqslant 2)$.

PROOF: $Z(D_{8n}) \simeq \{1, -1\}$ is characteristic, hence $f(-1) = -1$ for any $f \in$ Aut D_{8n}.

From this we have the following:

Proposition 1.6—Let G be a normal cubic extension of dihedral group H such that for $|H|$ admissible coset representative systems, f, is trivial. Then G is the alternating group A_4 of order 12 and $H \simeq \overset{2}{\oplus} Z_2$.

PROOF: In view of Theorem 1.2, H has to be nilpotent and the only dihedral groups which are nilpotent are powers of two. Further, in view of the discussion after Theorem 1.1, each element of $G - H$ is of order three. Next, Aut H contains no fixed-point-free automorphism if $|H| \geqslant 2^3$ in view of the Proposition 1.5 and the fact that Aut $D_8 \simeq D_8$. Obviously G can be written as semidirect product of H by Z_3, i.e. $G = H \underset{\phi}{\times} Z_3$ where $\phi \in$ Hom $(Z_3$, Aut $H)$. Represent $Z_3 = \{1, x, y\}$. Then $G - H = \{(h, \bar{x}): h \in H, \bar{x}$ is x or $y\}$. Then $(h, \bar{x})^3 = (h, \phi(\bar{x})\, h\, \phi(\bar{x})\, \phi(\bar{x})\, h, 1) = (1, 1)$. *i.e.*, $h\, f(h) f^2(h) = 1$ where $\phi(x) = f \in$ Aut H. Also H is not direct summand of G for $(|H|, 3) = 1$. Further $\phi(x) = f$ is non-trivial. $\phi(\bar{x})^3 = 1$ gives f is of order three. f will be fixed-point-free for $(|H|, 3) = 1$. This fact implies that $|H| \leqslant 4$. Further H contains an automorphism of order three, so $H \neq Z_2$. Hence the only possibility is that $H \simeq \overset{2}{\oplus} Z_2$. In this case Aut $H \simeq S_3$ and it contains fixed point automorphism $\phi(x)$ of order three. Thus $G \simeq \overset{2}{\oplus} Z_2 \underset{\phi}{\times} Z_3$ where $f = \phi(x) \in$ Aut $\overset{2}{\oplus} Z_2$, such that $hf(h) f^2(h) = 1$, which is A_4.

Thus we have characterized A_4.

Finally we have,

Proposition 1.7—Let G be a normal cubic extension of H such that for $|H|$ admissible coset representative systems, f, is trivial. Then H can never be Quaternian group or cyclic group of order $4n$ or Dihedral groups of order other than four.

PROOF: In view of the Theorem 1.2, Propositions 1.5 and 1.6 we only need to show that H cannot be quaternion group. Under the hypothesis of the proposition, H must contain an element of order three and it will be fixed-point-free for $(|H|, 3) = 1$. But the only quaternion group that contains an element of odd order is the Hamiltonian group of order eight. And the proof is complete in view of the Example 1.3.

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