UNIVALENT FUNCTIONS $f(z)$ FOR WHICH $zf'(z)$ IS α-SPIRAL-LIKE

by V. SINGH, *Department of Mathematics, Punjabi University, Patiala* 147002

and

P. N. CHICHRA, *Department of Physics, Punjabi University, Patiala* 147002

(*Received* 23 *May* 1975; *after revision* 27 *November* 1975)

Let $f(z) = z + a_2z^2 + \ldots$ be regular in $E = \{z/|z| < 1\}$ and let $|\alpha| < \pi/2$. It is shown that if $zf'(z)$ be α-spiral-like in E and $f''(0) = 0$, then $f(z)$ is univalent in E. A more general class of regular functions $f(z)$ for which

$$f'(z) = \left(\frac{s_1(z)}{z}\right)^{\lambda \cos \alpha \exp(-i\alpha)} \left(\frac{s_2(z)}{z}\right)^{\mu \cos \beta \exp(-i\beta)},$$

where $\lambda, \mu, \alpha, \beta$ are real, $|\alpha| < \pi/2$, $|\beta| < \pi/2$, and $s_1(z)$, $s_2(z)$ are normalized starlike functions has also been considered.

§1. A function $f(z)$ which is regular in the open unit disc $E = \{z/|z| < 1\}$ and satisfies the conditions $f(0) = 0 = f'(0) - 1$, and

$$\text{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0, \ z \text{ in } E, \qquad \ldots(1.1)$$

for some real number α, $|\alpha| < \pi/2$, is called an α-spiral-like function. We shall denote by $\mathcal{F}_\alpha$ the class of all α-spiral-like functions in E. Špaček (1933) showed that all α-spiral-like functions are univalent in E. Robertson (1969) considered the class $\mathcal{G}_\alpha$ of regular functions $f(z) = z + \ldots$ in E for which $zf'(z) \in \mathcal{F}_\alpha$. He proved that $f(z) \in \mathcal{G}_\alpha$ is univalent in E if $0 < \cos\alpha \leqslant 0{\cdot}2315\ldots$ and that $f(z) \in \mathcal{G}_\alpha$ is not univalent in E if $1/2 < \cos\alpha < 1$. Later Libera and Ziegler (1972) and Chichra (1975) improved upon the range of $\cos\alpha$ for which $f(z) \in \mathcal{G}_\alpha$ is univalent in E. All these authors used only Nehari's (1949) test of univalence and it has been shown by Chichra (1975) that if $f(z) \in \mathcal{G}_\alpha$, then $f(z)$ is univalent in E for $0 < \cos\alpha \leq (\sqrt{6} - \sqrt{2})/4 = 0{\cdot}2588\ldots$

Recently Ahlfors (1973) has obtained a new criterion of univalence. Using Ahlfors' criterion of univalence Pfaltzgraff (1975) has shown that $f(z) \in \mathcal{G}_\alpha$ is univalent in E for $0 < \cos\alpha \leq 1/2$. In this paper we show that functions $f(z) \in \mathcal{G}_\alpha$ which satisfy $f''(0) = 0$ are univalent in E for all α, $|\alpha| < \pi/2$.

If S^* denotes the class of normalized functions which are regular, univalent and starlike in E, then it is easy to see that $f(z) \in \mathcal{G}_\alpha$ if and only if there exists a function $s(z) \in S^*$ such that

$$f'(z) = (s(z)/z)^{\cos\alpha \exp(-i\alpha)}. \qquad \ldots(1.2)$$

Moulis (1975) studied some properties of the class of regular functions $f(z)$ for which

$$f'(z) = \left\{\left(\frac{s_1(z)}{z}\right)^{(k+2)/4}\left(\frac{s_2(z)}{z}\right)^{(2-k)/4}\right\}^{\cos\alpha\exp(-i\alpha)}, \quad k \geqslant 2, \ |\alpha| < \frac{\pi}{2},$$

for z in E, where $s_1(z)$ and $s_2(z)$ are in S^*. In §3 we improve upon and extend some of the results of Moulis to a wider class $G(\lambda, \mu, \alpha, \beta)$ of functions $f(z)$ which are regular in E and are normalized so that $f(0) = 0 = f'(0) - 1$, and for which

$$f'(z) = \left(\frac{s_1(z)}{z}\right)^{\lambda\cos\alpha\exp(-i\alpha)}\left(\frac{s_2(z)}{z}\right)^{\mu\cos\beta\exp(-i\beta)}, \qquad \ldots(1.3)$$

where $s_1(z)$ and $s_2(z) \in S^*$, and $\lambda, \mu, \alpha, \beta$ are real numbers with $|\alpha| < \pi/2$, $|\beta| < \pi/2$. Here

$$G(1, 0, \alpha, \beta) \equiv \mathcal{G}_\alpha \quad \text{and} \quad G\left(\frac{k+2}{4}, \frac{2-k}{4}, \alpha, \alpha\right)$$

with $k \geqslant 2$ is the class introduced by Moulis.

§2. Let $\mathcal{P}$ denote the class of regular functions $P(z) = 1 + c_1 z + \ldots$ which satisfy the condition Re $P(z) > 0$ for z in E.

The following test of univalence is due to Ahlfors (1975).

Lemma 1—Let $f(z)$ be regular in E and $f'(0) = 1$. Then $f(z)$ is univalent in E if for some c, $|c| \leqslant 1$, $c \neq -1$, and for all r, $0 \leqslant r = |z| < 1$, we have

$$\left|\frac{zf''(z)}{f'(z)} + \frac{cr^2}{1-r^2}\right| \leqslant \frac{1}{1-r^2}. \qquad \ldots(2.1)$$

Lemma 2—Let $P(z) \in \mathcal{P}$ and let $P'(0) = 0$. Then for all $r = |z|$, $0 \leqslant r < 1$, we have

$$\left|P(z) - \frac{1+r^4}{1-r^4}\right| \leqslant \frac{2r^2}{1-r^4}. \qquad \ldots(2.2)$$

Equality occurs in (2.2) for $P(z) = (1 + z^2)/(1 - z^2)$.

PROOF: Under the given conditions on $P(z)$, we can write

$$P(z) = \frac{1 - z\phi(z)}{1 + z\phi(z)},$$

where $\phi(z)$ is regular in E, $\phi(0) = 0$ and $|\phi(z)| < 1$ for z in E. Therefore by Schwarz lemma, we have

$$|\phi(z)| = \left|\frac{1 - P(z)}{z(1 + P(z))}\right| \leqslant r.$$

On simplifying this we obtain (2.2).

A more general form of Lemma 2 is given by Paul (1976).

Theorem 1—Let $f(z) \in \mathcal{G}_\alpha$ and $f''(0) = 0$. Then $f(z)$ is univalent in E for all α, $|\alpha| < \pi/2$.

PROOF: Let $f(z) \in \mathcal{G}_\alpha$. Then there exists a function $s(z) \in S^*$ such that (1.2) holds. Differentiating (1.2) and putting $zs'(z)/s(z) = P(z) \in \mathcal{P}$ we obtain

$$\frac{zf''(z)}{f'(z)} = \cos\alpha\, e^{-i\alpha}\,(P(z) - 1). \qquad \text{...(2.3)}$$

In view of Lemma 2 we find that the quantity $zf''(z)/f'(z)$ lies in the disc

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^4}{1-r^4}\cos\alpha\, e^{-i\alpha}\right| \leqslant \frac{2r^2\cos\alpha}{1-r^4}. \qquad \text{...(2.4)}$$

From this inequality we see that the value region of $zf''(z)/f'(z)$ lies in the disc

$$\left|\frac{zf''(z)}{f'(z)} + \frac{cr^2}{1-r^2}\right| \leqslant \frac{1}{1-r^2},$$

where $c = -\cos\alpha\, e^{-i\alpha}$. Since $|c| < 1$ for all α, $|\alpha| < \pi/2$, we conclude from Lemma 1 that $f(z)$ is univalent in E.

3. In this section we shall study some properties of the class $G(\lambda, \mu, \alpha, \beta)$.

Theorem 2—Let $f(z) \in G(\lambda, \mu, \alpha, \beta)$ and $\zeta \in E$. Then $F(z)$ defined by

$$\left.\begin{aligned} F'(z) &= \frac{f'\left(\dfrac{z+\zeta}{1+\bar{\zeta}z}\right)}{f'(\zeta)\,(1+\bar{\zeta}z)^{2\lambda\cos\alpha\exp(-i\alpha)+2\mu\cos\beta\exp(-i\beta)}} \\ F(0) &= 0, \end{aligned}\right\} \qquad \text{...(3.1)}$$

is also a member of $G(\lambda, \mu, \alpha, \beta)$.

This theorem is known for (i) $\mu = 0$, $\lambda = 1$ (Libera and Ziegler 1972) and (ii) $\lambda = (k+2)/4$, $\mu = (2-k)/4$, $\alpha = \beta$, $k \geqslant 2$ (Moulis 1972). The proof of this theorem follows from the fact (Libera and Ziegler 1972) that if $s(z) \in S^*$, then the function

$$\frac{\zeta z s\left(\dfrac{z+\zeta}{1+\bar{\zeta}z}\right)}{s(\zeta)\,(z+\zeta)\,(1+\bar{\zeta}z)}, \quad (|\zeta| < 1),$$

also belongs to S^*.

It is easy to prove that

Theorem 3—If $f(z) = z + a_2 z^2 + \dots$ is in $G(\lambda, \mu, \alpha, \beta)$, then

$$|a_2| \leqslant |\lambda| \cos\alpha + |\mu| \cos\beta. \qquad \dots(3.2)$$

This result is sharp.

Theorem 4—Let $f(z) \in G(\lambda, \mu, \alpha, \beta)$ and let

$$2(|\lambda| \cos\alpha + |\mu| \cos\beta) \leqslant 1. \qquad \dots(3.3)$$

Then $f(z)$ is univalent in E.

On putting $\lambda = 1$ and $\mu = 0$ in the above theorem we obtain a result of Pfaltzgraff (1975).

PROOF: If $f(z) \in G(\lambda, \alpha, \beta)$ and $F(z)$ is defined by (3.1), then $F(z) \in G(\lambda, \mu, \alpha, \beta)$, by Theorem 2, and it has the power series expansion.

$$F(z) = z + \frac{z^2}{z}\left\{(1 - |\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta}(\lambda \cos\alpha\, e^{-i\alpha} + \mu \cos\beta\, e^{-i\beta})\right\} + \dots \qquad \dots(3.4)$$

Therefore, from Theorem 3, we have for each r, $0 \leq |z| = r < 1$,

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1 - r^2}(\lambda \cos\alpha\, e^{-i\alpha} + \mu \cos\beta\, e^{-i\beta})\right| \leq \frac{2r}{1 - r^2}(|\lambda| \cos\alpha + |\mu| \cos\beta). \qquad \dots(3.5)$$

When (3.3) holds, the value region of $z f''(z)/f'(z)$ is contained in the disc

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1 - r^2}(\lambda \cos\alpha\, e^{-i\alpha} + \mu \cos\beta\, e^{-i\beta})\right| \leq \frac{1}{1 - r^2}.$$

Therefore, from Lemma 1, we conclude that if (3.3) holds, then $f(z) \in G(\lambda, \mu, \alpha, \beta)$ is univalent in E except when

$$2(\lambda \cos\alpha\, e^{-i\alpha} + \mu \cos\beta\, e^{-i\beta}) = 1. \qquad \dots(3.6)$$

But (3.3) and (3.6) are compatible only when $\alpha = 0 = \beta$, $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu = \frac{1}{2}$. However in this case it is easy to see that $f(z) \in G(\lambda, \mu, \alpha, \beta)$ is a convex function of order $\frac{1}{2}$. This completes the proof of Theorem 4.

Theorem 5—If $f(z) \in G(\lambda, \mu, \alpha, \beta)$, then $f(z)$ maps $|z| < r_0$ onto a convex domain where r_0 is the positive root of

$$1 - 2r(|\lambda| \cos\alpha + |\mu| \cos\beta) + r^2(2\lambda \cos^2\alpha + 2\mu \cos^2\beta - 1) = 0 \qquad \dots(3.7)$$

in (0, 1). This result is sharp when λ and μ are not both negative.

PROOF: If $f(z) \in G(\lambda, \mu, \alpha, \beta)$, then from (3.5) we obtain

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\geq 1+\frac{2r^2}{1-r^2}(\lambda\cos^2\alpha+\mu\cos^2\beta)-\frac{2r}{1-r^2}(|\lambda|\cos\alpha+|\mu|\cos\beta)$$
$$=\frac{1-2r(|\lambda|\cos\alpha+|\mu|\cos\beta)+r^2(2\lambda\cos^2\alpha+2\mu\cos^2\beta-1)}{1-r^2} \quad \text{...(3.8)}$$

Thus we see that $\operatorname{Re}(1+zf''(z)/f'(z))>0$ is satisfied for $|z|<r_0$, where r_0 is as stated in the Theorem. This proves that $f(z) \in G(\lambda, \mu, \alpha, \beta)$ maps $|z|<r_0$ onto a convex domain.

Now consider the function $f(z)$ given by

$$f(0)=0,$$
$$f'(z)=(1-ze^{i\theta})^{-2\lambda\cos\alpha\exp(-i\alpha)}(1-ze^{i\phi})^{-2\mu\cos\beta\exp(-i\beta)}, \quad \text{...(3.9)}$$

where

$$e^{i\theta}=\frac{r-e^{i\alpha}}{1-re^{i\alpha}},\ e^{i\phi}=\frac{r+e^{i\beta}}{1+re^{i\beta}} \quad \text{...(3.10)}$$

This function is in $G(\lambda, \mu, \alpha, \beta)$. Equality occurs in (3.8) for this function when λ and μ are of the opposite signs, when λ and μ are both non-negative, equality occurs in (3.8) for $f(z)$ given by (3.9) where now we choose

$$e^{i\theta}=\frac{r-e^{i\alpha}}{1-re^{i\alpha}},\ e^{i\phi}=\frac{r-e^{i\beta}}{1-re^{i\beta}} \quad \text{...(3.11)}$$

Thus we see that the result of Theorem 6 is sharp when λ and μ are not both negative.

On putting $\lambda=(k+2)/4$ and $\mu=(2-k)/4$, in the above theorem we obtain a result of Moulis (1972). But he was unable to show that his result was sharp.

§4. Libera and Ziegler (1972) had obtained sharp radius of close-to-convexity for $G(1, 0, \alpha, \beta)$. Also radius of close-to-convexity of $G\left(\frac{k+2}{4}, \frac{2-k}{4}, 0, 0\right)$ where $k \geq 2$ is known (Coonce and Ziegler 1972). The techniques used by these authors are similar to those employed by Krzyż (1962) to obtain the radius of close-to-convexity for the class of univalent functions. By employing similar techniques the radius of close-to-convexity of $G\left(\frac{k+2}{4}, \frac{2-k}{4}, \alpha, \alpha\right)$, where $k \geq 2$, can also be obtained. We state below, without proof, the sharp radius of close-to-convexity of this class of functions.

Theorem 6—If $\alpha \neq 0$, $k \geq 2$, $r_0 = 2\ (k \cos\alpha + (k^2 \cos^2\alpha - 4\ \cos 2\alpha)^{\frac{1}{2}})$, $\{r_0$ is the radius of close-to-convexity of $G\left(\frac{k+2}{4}, \frac{2-k}{4}, \alpha, \alpha\right)\}$, $r \in (r_0, 1)$,

$$x_0 = \left\{ \frac{\begin{array}{c} 8\sin^2\alpha\,(1+r^2)(1-r^2\cos 2\alpha) + k^2\cos^2\alpha\,(1-2r^2\cos 2\alpha + r^4) \\ - \{k^4\cos^4\alpha\,(1-2r^2\cos 2\alpha + r^4)^2 \\ - 32\ k^2r^2\cos^4\alpha\sin^2\alpha\,(1-r^4)\ (1-r^2\cos 2\alpha)\}^{\frac{1}{2}} \end{array}}{8r^2\sin^2\alpha\ (4\sin^2\alpha + k^2\cos^2\alpha)} \right\}^{\frac{1}{2}},$$

$$\theta_0 = 2\ \text{arc}\ \cos\ x_0,\ 0 < \theta_0 < \pi,$$

and

$$\Delta(r) = \theta_0 + 2\cos^2\alpha\ \text{arc}\ \tan\left(\frac{r^2\sin\theta_0}{1 - r^2\cos\theta_0}\right)$$

$$- k\cos^2\alpha\,\text{arc}\sin\left[r\cos\alpha\left\{\frac{2(1-\cos\theta_0)}{1-2r^2\cos\theta_0 + r^4}\right\}^{\frac{1}{2}}\right]$$

$$+ \frac{k}{2}\ \sin\ 2\alpha\log\ [\{1 - 2r^2\ (\cos\ \theta_0\ \sin^2\alpha\ + \cos^2\alpha) + r^4\}^{\frac{1}{2}}$$

$$- r\sin\alpha\,\{2\,(1-\cos\theta_0)\}^{\frac{1}{2}}]$$

$$- \frac{k}{2}\sin\ 2\alpha\ \log\ (1-r^2);$$

then the radius of close-to-convexity of $G\left(\frac{k+2}{4}, \frac{2-k}{4}, \alpha, \alpha\right)$ is the unique root of the equation $\Delta(r) = -\pi$.

In the proof of the above theorem the following lemma is used.

Lemma 3—If $f(z) \in G\left(\frac{k+2}{4}, \frac{2-k}{4}, \alpha, \alpha\right)$, $k \geq 2$, and $|z| = r < 1$, then

$$\arg f'(z) \geq -k\cos^2\alpha\ \text{arc}\ \sin\ (r\cos\alpha) - ((k-2)/4)\ \sin\ 2\alpha\log\ (1-r^2)$$
$$+ (k/2)\ \sin\ 2\alpha\log\ ((1-r^2\cos^2\alpha)^{\frac{1}{2}} - r\ \sin\ \alpha).$$

The proof of the above lemma follows easily on using corresponding known result for spiral-like functions (*see* Libera and Ziegler 1972, Lemma 4).

5. *Concluding remarks*— Functions $f(z) \in \mathcal{G}_\alpha$ satisfy (2.2). However, result of Theorem 1 remains valid if, instead of satisfying (2.2), $f(z)$ satisfies the condition

$$\frac{zf''(z)}{f'(z)} = \cos\ \alpha\ \exp\ (i\beta)\,(P(z)-1),$$

where β is any real number independent of α.

REFERENCES

Ahlfors, L. V. (1973). Sufficient conditions for quasiconformal extension. *Proc.* 1973 *Conf. Univ. of Maryland, Ann. Math. Studies*, **79**, 23–29.

Chichra, P. N. (1975). Regular functions $f(z)$ for which $z\ f'(z)$ is α-spiral-like. *Proc. Am. math. Soc.*, **48**.

Coonce, H. B., and Ziegler, M. R. (1972). The radius of close-to-convexity of functions of bounded boundary rotation. *Proc. Am. math. Soc.*, **35**, 207–10.

Krzyż, J. (1962). The radius of close-to-convexity within the family of univalent functions. *Bull. Acad. Polon. Math. Soc. Ser. Sci. Math. Astro. Phys.*, **10**, 201–4.

Libera, R. J., and Ziegler, M. R. (1972). Regular functions $f(z)$ for which $zf'(z)$ is α-spiral. *Trans. Am. math. Soc.*, **166**, 361–70.

Moulis, E. J. (Jr.) (1972). A generalization of univalent functions with bounded boundary rotation. *Trans. Am. math. Soc.*, **174**, 369–81.

Nehari, Z. (1949). The Schwarzian derivative and Schlicht functions. *Bull. Am. math. Soc.*, **55**, 545–51.

Pfaltzgraff, J. A. (1975). Univalence of the integral of $f'(z)^{\lambda}$. *Notices Am. math. Soc.*, **22**, No. 1, 720–30–27.

Robertson, M. S. (1969). Univalent functions $f(z)$ for which $zf'(z)$ is spiral-like. *Mich. Math. J.*, **16**, 97–101.

Špaček, L. (1933). Přispěvek k teorii funkci prostych, *Časopis Pěst. Mat. Fys.*, **62**, 12–19.

Paul, Surinder (1976). An extremal problem for starlike functions. *Indian J. pure appl. Math.* (To appear).