EXCEPTIONAL VALUES OF ENTIRE CHARACTERISTIC FUNCTIONS

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It is shown that a class of entire characteristic functions cannot have any finite exceptional value in the sense of Borel. Some extensions, with different types of exceptional values, have been proved.

1. INTRODUCTION

Let $f(z)$ be an entire function and denote by $M(r, f)$ the maximum modulus of $f(z)$ on $|z| = r$ and by $n(r, \alpha)$ the number of zeros of $f(z) - \alpha$ in $|z| \leqslant r$. Let E denote the set of positive non-decreasing functions ϕ such that

$$\int_A^{\infty} \frac{dx}{x\,\phi(x)}$$

is convergent. A value $\alpha\,(0 \leqslant |\alpha| \leqslant \infty)$ is said to be an exceptional value E (e.v.E.) of $f(z)$ if

$$\liminf_{r\to\infty} \log M(r, f)/\big(n(r, \alpha)\,\phi(r)\big) > 0$$

for some $\phi \in E$ (cf. Shah 1951).

If

$$\limsup_{r\to\infty} \log^{+} n(r, \alpha)/\log r = \rho_1(\alpha) < \rho,$$

where ρ denotes the order of $f(z)$, then α is said to be an exceptional value in the sense of Borel (e.v.B.) of $f(z)$. It is known that if $f(z)$ is of finite order ρ and has a finite e.v.E. then ρ is necessarily a positive integer and $f(z)$ is of perfectly regular growth order ρ. If α is an e.v.B. then it is also an e.v.E. but the converse is not true (Shah 1951).

An entire characteristic function (c.f.) is a function f of the form

$$f(z) = \int_{-\infty}^{\infty} e^{izt}\,dF(t) \qquad \ldots(1.1)$$

where $F(t)$ is a non-decreasing and right-continuous on $(-\infty, \infty)$ and is such that $F(-\infty) = 0$, $F(+\infty) = 1$ [F is called a probability distribution function (see Lukacs 1970, pp. 1–2)], and such that for every $k > 0$

$$\int_{-\infty}^{\infty} e^{k|t|}\,dF(t) < \infty.$$

Marcinkiewicz (1938) (see also Lukacs 1958, 1970) proved, for functions of finite order, the following:

Theorem 1—An entire function of finite order $\rho > 2$ whose exponent of convergence of zeros $\rho_1(0)$ is less than ρ cannot be a characteiistic function.

In this paper we improve and extend this theorem. In what follows we suppose that m is zero or a positive integer, α and $\alpha_j \in \mathbb{C}$, $Q(z)$ denotes a polynomial, $P(z)$ denotes a canonical product (c.p.) and $f(z)$ an entire function.

Theorem 2—Let $\alpha \in \mathbb{C}$, ρ an integer greater than two, $Q(z)$ a polynomial of (exact) degree ρ, $P(z)$ a canonical product of finite order and such that

$$\liminf_{r\to\infty} \log M(r, P)/r^{\rho} = 0. \qquad \ldots(1.2)$$

Then

$$f(z) = \alpha + z^m \exp\big(Q(z)\big) P(z) \qquad \ldots(1.3)$$

is an entire function of finite order and is not a characteristic function.

Corollary 1—A characteristic entire function of finite order $\rho > 2$ cannot have a finite exceptional value E.

Corollary 2—Let $f(z)$ be an entire function of finite order $\rho > 2$. If there exists a complex number α such that $\rho_1(\alpha) < \rho$ then $f(z)$ cannot be a characteristic function.

Remarks: (i) The functions f and P, in (1.3), need not be of order ρ. (ii) It is clear that given $f(z)$, the polynomial $Q(z)$, the c.p. $P(z)$ and the integer m in (1.3) (and in (1.4)) will depend on α. (iii) In Theorems 2–4 we do not list obvious necessary conditions for $f(z)$ to be a c.f. Thus, for instance, if $f(0) \neq 1$, then $f(z)$ cannot be a c.f.

Theorem 3—Let an entire function f be defined by

$$f(z) = \alpha + z^m \{\exp(\alpha_1 z^2 + \alpha_2 z + \alpha_3)\} P(z) \qquad \ldots(1.4)$$

where $P(z)$ is the c.p. of finite order and such that

$$\liminf_{r\to\infty} \frac{\log M(r, P)}{r^{\rho}} = 0. \qquad \ldots(1.5)$$

Suppose that one of the following five conditions is satisfied:

(a) Im $\alpha_1 \neq 0$, $\rho = 2$;

(b) Re $\alpha_1 > 0$, $\rho = 2$;

(c) Re $\alpha_1 < 0$, $|\alpha| > 1$, $\rho = 2$;

(d) $\alpha_1 = 0$, Re $\alpha_2 \neq 0$, $\rho = 1$;

(e) $\alpha_1 = 0$, Re $\alpha_2 = 0$, $\rho = 1$, and (i) either $P(z)$ is not a constant or (ii) $P(z)$ a constant and $m > 0$. Then $f(z)$ cannot be a c.f.

Theorem 4—Let an entire function f be defined by

$$f(z) = \{\exp(\alpha_1 z^2 + \alpha_2 z)\} \; P(z) \qquad \ldots(1.6)$$

where $P(z)$ is the c.p. of genus $p \, (p < \infty)$. Suppose that one of the following conditions is satisfied:

(*a*) $\operatorname{Im} \alpha_1 \neq 0$;

(*b*) $\operatorname{Re} \alpha_2 \neq 0$;

(*c*) $\operatorname{Re} \alpha_1 \geqslant 0, \; p \geqslant 2$;

(*d*) $\operatorname{Re} \alpha_1 > 0, \; p = 0$ or 1;

(*e*) $\operatorname{Re} \alpha_1 = 0, \; p = 0$ and $P(z)$ not a constant. Then $f(z)$ cannot be a c.f.

Remarks : (i) The c.f. of the Poisson distribution (Lukacs 1970, p. 18) shows that the condition that ρ be finite in Theorem 2 is necessary. (ii) If we consider a function of order 2 and having no zeros we get from Theorem 4 that $f(z) = \exp\{(A + iB) z^2 + (C + iD) z\}$ cannot be a c.f. unless $B = 0 = C$ and $A \leqslant 0$. Since $f(z)$ is of order 2 we must have $A < 0$ and we get $f(z) = \exp(Az^2 + iDz)$, $A < 0$, D real, which is the c.f. of the Normal distribution (Lukacs 1972, p. 18). Similarly, the condition (*b*) of Theorem 4 shows that the only function, of order one and having no zeros, which can qualify to be a c.f. is $f(z) = \exp(iDz)$, D real and non-zero; and this function is the c.f. of the degenerate distribution (Lukacs 1970, p. 18). (iii) There exist char. functions of the form (1.6) and having an infinity of zeros. Take for instance, $f(z) = \exp(-z^2/2) (\sin z/z)$. (iv) The c.f. of the Rectangular distribution has order 1, $\alpha = 0 = m = \alpha_1 = \operatorname{Re} \alpha_2 = \alpha_3$ but the c.p. $P(z)$ does not satisfy conditions (*e*) and (1.5) of Theorem 3.

2. Proof of Theorem 2

We require the following:

Lemma—Let $F(z)$ be an entire function of finite order. Let $\rho > 0$, $k > 1$ and suppose that

$$\liminf_{r \to \infty} \frac{\log M(r, F)}{r^\rho} = 0. \qquad \ldots(2.1)$$

Given $\varepsilon > 0$, there exists a sequence $\{r_j\}_{j=1}^{\infty}$, $r_j \uparrow \infty$, such that

$$\log m(r, F) > -\varepsilon H r^\rho, \log M(r, F) < \varepsilon k^{2\rho} r^\rho, r = r_j, \; j = 1, 2, \ldots \qquad \ldots(2.2)$$

Here

$$m(r, F) = \min_{|z| = r} |F(z)| \text{ and } H = H(k).$$

Proof : We have for a sequence $\{R_j\}_{j=1}^{\infty}$

$$M(R_j, F) < \exp(\varepsilon R_j^\rho)$$

Now there is at least one r in $\left[\frac{R_j}{k^2}, \frac{R_j}{k}\right]$ such that (Valiron 1949, p. 89)

$$m(r, F) > \{M(kr, F)\}^{-H_1}$$

and

$$M(kr, F) < M(R_j, F) < \exp(\varepsilon R_j^{\rho}) < \exp(\varepsilon k^{2\rho} r^{\rho}).$$

Further

$$\log m(r, F) > -H_1 \log M(kr, F) > -H_1 \varepsilon k^{2\rho} r^{\rho} = -\varepsilon H r^{\rho},$$

and the lemma is proved.

Suppose if possible that $f(z)$ is a c.f. Then $f(z)$ is of the form

$$f(z) = \int_{-\infty}^{\infty} e^{izt}\, dF(t). \qquad \text{...(2.3)}$$

We write $z = x + iy = re^{i\theta}$. Then from (2.3)

$$|f(x)| \leqslant 1, -\infty < x < \infty, \qquad \text{...(2.4)}$$

$$|f(x+iy)| \leqslant f(iy), -\infty < x < \infty, -\infty < y < \infty. \qquad \text{...(2.5)}$$

Let $Q(z) = (A + iB) z^{\rho} + Q_1(z)$ where $Q_1(z)$ is a polynomial of degree not exceeding $\rho - 1$. Write $A + iB = Re^{i\phi}$, $-\pi < \phi \leqslant \pi$. We consider three cases (i) $A > 0$, (ii) $A < 0$ and (iii) $A = 0$ separately. In the sequel, $j > j_0(\varepsilon)$ will mean that j is sufficiently large. The value j_0 will, in general, vary.

(i) $A > 0$. We have for $x = r_j$, $j > j_0$,

$$|f(x)| \geqslant x^m \exp(Ax^{\rho}) \,|e^{Q_1(x)}|\, |P(x)| - |\alpha|$$
$$\geqslant x^m \exp\{Ax^{\rho} - \varepsilon x^{\rho} - H\varepsilon x^{\rho}\} - |\alpha|$$

and the last expression tends to ∞ as $r_j \to \infty$ contradicting (2.4).

(ii *a*) $A < 0$, ρ even. If $B \neq 0$ we choose $x = r_j \cos\theta_0$, $y = r_j \sin\theta_0$, $\theta_0 = -\phi/\rho$, and then for $j > j_0$

$$|f(iy)| \leqslant |\alpha| + |y|^m \exp\{(|A| + \varepsilon)|y|^{\rho}\} M(r_j, P)$$
$$< \exp\{(|A| + 2\varepsilon + \varepsilon k^{2\rho}) r^{\rho}\}, \qquad r = r_j;$$
$$|f(x+iy)| \geqslant -|\alpha| + r^m \exp\{(R - 2\varepsilon - H\varepsilon) r^{\rho}\}$$
$$\geqslant \exp\{(R - 3\varepsilon - H\varepsilon) r^{\rho}\}, \qquad r = r_j.$$

Using (2.5) we get a contradiction since $R > |A|$. If $B = 0$, then $\phi = \pi$, $\theta_0 = -\pi/\rho$ and since $\rho \geqslant 3$,

$$|y|^{\rho} = |r_j \sin\theta_0|^{\rho} \leqslant r_j^{\rho}\left(\sin\frac{\pi}{\rho}\right)^{\rho} < \frac{3}{4} r_j^{\rho}.$$

Hence

$$|f(iy)| \leqslant |\alpha| + r_j^m \exp\{(\tfrac{3}{4}|A| + \varepsilon) r_j^{\rho}\} M(r_j, P),$$
$$|f(x+iy)| \geqslant -|\alpha| + r_j^m \exp\{(|A| - 2\varepsilon - H\varepsilon) r_j^{\rho}\}.$$

Using (2.2) and (2.5) we get a contradiction since $A \neq 0$.

(ii *b*) $A < 0$, ρ odd. Here ϕ and θ_0 are as in (ii *a*) and $r = r_j$, $j > j_0$. Then

$$| f(iy) | \leqslant | \alpha | + r^m \, | \exp ((A + iB)(iy)^\rho) | \exp \{2\varepsilon k^{2\rho} r^\rho\}$$
$$\leqslant \exp \{(| B | + 3\varepsilon \, k^{2\rho}) \, r^\rho\},$$
$$| f(x + iy) | \geqslant - | \alpha | + r^m \, | \exp (Re^{i\phi} \, r^\rho \, e^{i\rho\theta_0}) | \exp ((-\varepsilon - H\varepsilon) \, r^\rho)$$
$$> \exp \{(R - 2\varepsilon - H\varepsilon) \, r^\rho\}$$

and we get a contradiction since $R > | B |$.

(iii *a*) $A = 0$, ρ odd. Hence $\rho \geqslant 3$, $B \neq 0$. We have $\phi = \frac{\pi}{2}$ if $B > 0$, $\phi = -\frac{\pi}{2}$ if $B < 0$,

$$\theta_0 = -\phi/\rho;$$

$$| y |^\rho = r^\rho \, | \sin \theta_0 |^\rho = r^\rho \left\{ \sin \frac{\pi}{2\rho} \right\}^\rho \leqslant \frac{r^\rho}{8}, \quad r = r_j.$$

Hence, for $r = r_j$, $j > j_0$,

$$| f(iy) | \leqslant | \alpha | + r^m \exp \{(| B |/8 + \varepsilon) \, r^\rho\}$$
$$| f(x + iy) | \geqslant - | \alpha | + r^m \exp \{(| B | - H\varepsilon - 2\varepsilon) \, r^\rho\}$$

and we have a contradiction since $B \neq 0$.

(iii *b*) $A = 0$, ρ even. As before $\phi = \frac{\pi}{2}$ if $B > 0$, $-\frac{\pi}{2}$ if $B < 0$ and $\theta_0 = -\phi/\rho$. Hence for $r = r_j$, $j > j_0$,

$$| f(iy) | \leqslant | \alpha | + r^m \, | \exp (iB \, (iy)^\rho) | \; | (P \, (iy)) |$$
$$\leqslant | \alpha | + r^m \exp (\varepsilon r^\rho)$$
$$| f(x + iy) | \geqslant - | \alpha | + r^m \exp \{(| B | - H\varepsilon - 2\varepsilon) \, r^\rho\}$$

and we have a contradiction since $B \neq 0$. Hence f cannot be a c.f. The theorem is proved.

3. Corollaries and Example

Corollary 1—Let $f(z)$ be of order $\rho > 2$ and suppose that it has α as an e.v.E. Then $f(z)$ can be written in form (1.3) and we have two possibilities: (i) $\rho_1(\alpha) < \rho$, (ii) $\rho_1(\alpha) = \rho$, $p(\alpha) = \rho - 1$ (see Shah 1951, p. 230). In either case the hypotheses of Theorem 2 are satisfied and the corollary follows.

Corollary 2—This follows immediately from Theorem 1 of Shah (1951).

We now construct a function f which is of the form (1.3) and the c.p. P satisfies (1.2) but it does not have α as an e.v.E.

Example—Let ρ be a positive integer and let $\{a_n\}_1^\infty$ be a non-decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} 1/(na_n^\rho) = \infty$. Let $P(z)$ be the c.p. formed with simple zeros at $\{n^{1/\rho} a_n\}$ and $\{\exp(i\pi/\rho)\ n^{1/\rho} a_n)\}$ ($n = 1, 2, \ldots$). Then $P(z)$ is of genus ρ and (1.2) is satisfied (Boas 1954, p. 27). Let f be defined by (1.3) where Q is a polynomial of degree ρ and P is the c.p. defined just now. Then $f(z)$ is an entire function of order ρ. If $\rho > 2$, or $Q(z) = \alpha_1 z^2 + \alpha_2 z + \alpha_3$ and $\rho = 2$ if $\alpha_1 \neq 0$, $\rho = 1$ otherwise, and one of the conditions (*a*) to (*e*) is satisfied, then f is not a c.f. If we now take $a_n = \{\log(n+1)\}^{1/\rho}$, we see that α is not an e.v.E.

4. Proof of Theorem 3

The proof is similar to that of Theorem 2 and is briefly sketched.

(*a*), (*b*) Here $\rho = 2$. Now note that we utilized the condition $\rho > 2$, when ρ is even, only in that part of case (ii *a*) where A $(= \text{Re}\ \alpha_1) < 0$ and $B = 0$.

(*c*) Here $\rho = 2$ and $|f(x) - \alpha| \to 0$ as $x \to +\infty$, and since $|\alpha| > 1$, $|f(x)| > 1$ for all large x. Now we use (2.4) to show that f cannot be a c.f.

(*d*) Here $\rho = 1$ and we utilized $\rho > 1$, when ρ is odd, only when A $(= \text{Re}\ \alpha_2) = 0$.

(*e*) Write $\alpha_2 = c + iD$. Then

$$f(z) = \alpha + z^m \exp(iDz + \alpha_3)\, P(z). \qquad \ldots(4.1)$$

(i) If $f(z)$ is a c.f. then $|f(x)|$ is bounded and so are $|f(x) - \alpha|$ and $|(f(x) - \alpha)/x^m|$ (consider the cases $|x| \geqslant 1$ and $|x| < 1$ separately). This implies $|P(x)|$ bounded and so (1.5) shows that $\log M(r, P) = o\ (r)$ (cf. Boas 1954, p. 97) and hence $P(z)$ must reduce to a constant (Boas 1954, p. 84). This contradicts our hypothesis.

(ii) If $m > 0$ and $P(z)$ a constant, then (4.1) shows that $|f(x)|$ is not bounded leading again to a contradiction. The proof of Theorem 3 is complete.

5. Proof of Theorem 4

(*a*), (*b*) Write $\alpha_1 = A + iB$, $\alpha_2 = C + iD$. If $f(z)$ is a c.f. then

$$f(z) = \overline{f(-\bar{z})}.$$

By considering the zeros we have $P(z) = \overline{P(-\bar{z})}$ and consequently $\exp(\alpha_1 z^2 + \alpha_2 z) = \exp(\bar{\alpha}_1 z^2 - \bar{\alpha}_2 z)$. This gives $B = 0 = C$ and we get a contradiction with hypothesis (*a*) or (*b*).

(*c*) We have

$$f(z) = \exp\{(A + iB)\, z^2 + (C + iD)\, z\}\, P(z).$$

Suppose $f(z)$ is a c.f. Then $B = 0 = C$. If $A \geqslant 0$ then $\exp(-Az^2 - iDz)$ is a c.f. and so $P(z)$ is a c.f. (Lukacs 1970, p. 38). Now, if $p \geqslant 2$, then $P(z) = 1 + o(|z|^2)$ as $z \to 0$ and we get a contradiction (Lukacs 1970, p. 68).

(*d*) Since

$$|f(x)| = \exp(Ax^2 + Cx) P(x),$$

(2.2) shows that $|f(x)|$ is unbounded as $|x| \to \infty$.

(*e*) Suppose $f(z)$ is a c.f. Then

$$f(z) = e^{iDz} P(z)$$

and so $P(z)$ is a c.f. But $p = 0$ and so $\log M(r, P) = o(r)$. Consequently $P(z)$ must reduce to a constant, contradicting the hypothesis.

This completes the proof.

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