SOME PLANE WAVE-LIKE SOLUTIONS OF WEAKENED FIELD EQUATIONS IN A GENERALIZED PERES SPACE-TIME

by K. B. LAL *and* ANIRUDH PRADHAN, *Department of Mathematics, University of Gorakhpur, Gorakhpur* 273001

(*Received* 29 *March* 1976)

In the sense of Takeno, plane wave-like solutions of a set of five weakened field equations have been investigated in a generalized Peres space-time. It has been shown that the solutions obtained by Lal and Pandey are derivable from those obtained in this paper.

1. INTRODUCTION

Einstein field equation of general relativity in vacuum is given by

$$R_{ij} = 0. \qquad ...(1.1)$$

Several field equations alternative to (1.1) but weaker than it have been suggested by various authors like Kilmister and Newmann (1961), Rund (1964, 1966, 1967), Buchdahl (1962), Duplessis (1965) and Pechlaner and Sexl (1966). They are weaker in the sense that each such field equation admits (1.1) as a subclass of solution. Such field equations have been called weakened field equations in vacuum. The physical aspects of weakened field equations are not well established though many researchers (see for example, Thompson 1963; Lovelock 1967*a*, *b*; Swami 1970; Lal and Singh 1973; Lal and Pandey 1975; Pandey 1975) have found the solutions in the hope to give a useful interpretation later. Thompson (1963) has made detailed study of weakened field equations and showed conclusively that weakened field equations are too weak. Lovelock (1967*a*, *b*) has solved a set of five weakened field equations, namely

$$J_{ikl} \equiv R^{j}{}_{ikl;j} = 0, \qquad ...(1.2)$$

$$G_{jk} \equiv (-g)^{\frac{1}{2}} [g^{ih} R_{kj;ih} - g^{ih} R_{ij;kh} + (1/6) R_{;kj} - (1/6) g_{jk} g^{ih} R_{;ih} - R^{ih} C_{jhik} + (R/6) g^{ij} C_{jhik}] = 0, \qquad ...(1.3)$$

with properties (*a*) $G_{jk} = G_{kj}$ and (*b*) $G^{j}{}_{k;j} = 0$,

$$E^{hk} \equiv (-g)^{\frac{1}{2}} [g^{hj} g^{ki} \{2R_{jlim} R^{ml} + g^{ml} R_{ij;lm} - R_{;ij}\} - (1/2) g^{hk} \{R^{l}{}_{m} R_{l}{}^{m} - g^{lm} R_{;lm}\}] = 0, \qquad ...(1.4)$$

with properties (*a*) $E^{hk} = E^{kh}$ and (*b*) $E^{hk}_{;h} = 0$,

$$\mathscr{E}^{rs} = (-g)^{\frac{1}{2}} [(g^{rs} g^{tu} - (1/2)\, g^{rt} g^{su} - (1/2)\, g^{ru} g^{st})\, R_{,ut} + R\,(R^{sr} - (1/4)\, g^{sr} R)] = 0 \qquad \ldots(1.5)$$

with properties (*a*) $\mathscr{E}^{rs} = \mathscr{E}^{sr}$ and (*b*) $\mathscr{E}^{rs}_{;r} = 0$,

$$H_k{}^{ij} \equiv R^{ij}_{;k} = 0, \qquad \ldots(1.6)$$

and has obtained static spherically symmetric solutions. Lovelock (1967*a*) has shown that these weakened field equations possess solutions which correspond to an isolated mass at origin which repels test particles. Due to this contradiction with experiment his solutions are highly unphysical. Here C_{jhik} is the Weyl curvature tensor defined by

$$C_{jhik} = R_{jhik} - (1/2)\,(R_{jk}\, g_{hi} - R_{hi}\, g_{jk} - R_{ji}\, g_{hk} + R_{hk}\, g_{ji}) + (R/6)\,(g_{ij}\, g_{hk} - g_{hi}\, g_{jk}), \qquad \ldots(1.7)$$

and a semicolon followed by an index denotes covariant differentiation.

Recently Lal and Pandey (1975) have found out the wave-like solutions of these weakened field equations in Peres space-time. In this paper we propose to obtain the plane wave-like solutions of weakened field equations (1.2)–(1.6) in a generalized Peres space-time, represented by the metric

$$ds^2 = -A\, dx^2 - B\, dy^2 - (1 - E)\, dz^2 - 2E\, dz\, dt + (1 + E)\, dt^2 \qquad \ldots(1.8)$$

where $A = A(x, Z)$, $B = B(y, Z)$ and $E = E\,(x, y, Z)$; $Z = (z - t)$. It is not difficult to verify that metric (1.8) admits a parallel null vector field and is also a gravitational null field, that conditions which characterize Peres space-time (Takeno 1962). Accordingly, the space-time represented by the metric (1.8) has been called a 'generalized Peres space-time'.

2. Components of $\left\{ {k \atop ij} \right\}$, R_{ijkl} and R_{ij}

The non-vanishing components of g^{ij} corresponding to the metric (1.8) are given by

$$g^{11} = -1/A, \quad g^{22} = -1/B, \quad g^{33} = -(1 + E), \quad g^{44} = 1 - E,$$
$$g^{34} = g^{43} = -E, \qquad \ldots(2.1)$$

while the non-vanishing components of the Christofell symbols of second kind $\left\{ {k \atop ij} \right\}$ are

$$\left\{ {1 \atop 11} \right\} = A_x/2A, \quad \left\{ {3 \atop 11} \right\} = \left\{ {4 \atop 11} \right\} = -\frac{1}{2}\bar{A}, \quad \left\{ {2 \atop 22} \right\} = B_y/2B$$

$$\left\{ {3 \atop 22} \right\} = \left\{ {4 \atop 22} \right\} = -\frac{1}{2}\bar{B}, \quad \left\{ {1 \atop 13} \right\} = -\left\{ {1 \atop 14} \right\} = \bar{A}/2A, \quad \left\{ {a \atop 13} \right\}$$

$$= -\left\{ {a \atop 14} \right\} = -\frac{1}{2}E_a$$

$$\left\{ {2 \atop 23} \right\} = - \left\{ {2 \atop 24} \right\} = \bar{B}/2B, \quad \left\{ {a \atop 23} \right\} = - \left\{ {a \atop 24} \right\} = - \frac{1}{2} E_y$$

$$\left\{ {1 \atop 33} \right\} = - \left\{ {1 \atop 34} \right\} = \left\{ {1 \atop 44} \right\} = E_x/2A, \quad \left\{ {2 \atop 33} \right\} = - \left\{ {2 \atop 34} \right\}$$

$$= \left\{ {2 \atop 44} \right\} = E_y/2B \qquad ...(2.2)$$

$$\left\{ {a \atop 33} \right\} = - \left\{ {a \atop 34} \right\} = \left\{ {a \atop 44} \right\} = - \frac{1}{2} \bar{E}, \quad (a = 3, 4).$$

The non-vanishing components of the curvature tensor are given by

$$\left.\begin{aligned} R_{1313} &= - R_{1314} = R_{1414} = \alpha \\ R_{1323} &= - R_{1324} = - R_{1423} = R_{1424} = \tfrac{1}{2} E_{xy} \\ R_{2323} &= - R_{2324} = R_{2424} = \beta \end{aligned}\right\} \qquad ...(2.3)$$

where

$$\alpha = \tfrac{1}{2} [\bar{\bar{A}} - E_{xx} - \tfrac{1}{2} (\bar{A}^2 - A_x E_x)/A]$$

$$\beta = \tfrac{1}{2} [\bar{\bar{B}} - E_{yy} - \tfrac{1}{2} (\bar{B}^2 - B_y E_y)/B].$$

The surviving components of the Ricci tensor R_{ij} obtained from (2.3) on contraction with the help of (2.1) are given by

$$R_{33} = - R_{34} = R_{44} = \varepsilon, \qquad ...(2.4)$$

where $\varepsilon = \alpha/A + \beta/B$. Here and elsewhere the lower suffixes x and y attached with any function denote partial differentiation with respect to x and y respectively while an overhead bar stands for partial differentiation with respect to Z. Also for the metric (1.8)

$$(a)\ R = 0, \quad (b)\ g = |g_{ij}| = - AB, \quad (c)\ R_m^{\ l} R_l^{\ m} = 0, \qquad ...(2.5)$$

3. Solutions of Weakened Field Equations (1.2)–(1.6)

We shall state our results in the form of theorems which generalize the recent results of Lal and Pandey (1975).

Theorem 1—A necessary and sufficient condition that g_{ij} given by (1.8) be a solution of weakened field equation (1.2) is

$$\varepsilon_x = 0 \quad \text{and} \quad \varepsilon_y = 0. \qquad ...(3.1)$$

Proof : The Bianchi identities which hold in a generalized Riemannian space, on first contraction, give

$$R^h_{\ ijk;h} + R_{ik;j} - R_{ij;k} = 0. \qquad ...(3.2)$$

From (1.2) and (3.2) we find that

$$R^h_{\ ijk;h} = 0 \Leftrightarrow R_{ij;k} - R_{ik;j} = 0.$$

Thus the form (1.2) of weakened field equation is equivalent to

$$R_{ij;k} - R_{ik;j} = 0. \qquad \ldots(3.3)$$

Substituting the components of R_{ij} from (2.4) and that of $\left\{ {k \atop ij} \right\}$ from (2.2) in (3.3) we find the required condition (3.1). Conversely, if (3.1) holds (3.3) is identically satisfied. This proves the theorem.

Theorem 2—A necessary and sufficient condition that g_{ij} given by (1.8) be a solution of weakened field equation (1.3) is

$$(\varepsilon_{xx} - \varepsilon_x A_x/2A)/A + (\varepsilon_{yy} - \varepsilon_y B_y/2B)/B = 0. \qquad \ldots(3.4)$$

PROOF : With the aid of (1.7), (2.5*a*), (2.5*b*), equation (1.3) becomes

$$G_{jk} = (AB)^{\frac{1}{2}} [g^{ih} \{R_{kj;ih} - R_{ij;kh}\} - R^{ih} \{R_{jhik} - \tfrac{1}{2} (R_{kj} g_{hi} - R_{hi} g_{jk} - R_{ji} g_{hk} + R_{hk} g_{ji})\}] = 0. \qquad \ldots(3.5)$$

Substituting the components of R_{ijkl}, R_{ij} and g^{ij} from (2.3), (2.4) and (2.1) into (3.5) we find that equations $G_{jk} = 0$ are satisfied for all values of j and k except for j, $k = 3$, 4 which on simplification give the required condition (3.4). Conversely, if (3.4) holds we see that all equations $G_{jk} = 0$ are identically satisfied. Hence the theorem.

Theorem 3—A necessary and sufficient condition that g_{ij} given by (1.8) be a solution of weakened field equation (1.4) is

$$(\varepsilon_{xx} - \varepsilon_x A_x/2A)/A + (\varepsilon_{yy} - \varepsilon_y B_y/2B)/B = 0. \qquad \ldots(3.6)$$

PROOF : In view of (2.5), equation (1.4) reduces to

$$E^{hk} = (AB)^{\frac{1}{2}} \; [g^{hj} g^{ki} (2R_{jlim} R^{ml} + g^{ml} R_{ij;lm})] = 0, \qquad \ldots(3.7)$$

for the metric (1.8). By virtue of (2.1), (2.3) and (2.4) we find that (3.7) is satisfied if and only if

$$E^{33} = E^{34} = E^{44} = (\varepsilon_{xx} - \varepsilon_x A_x/2A)/A + (\varepsilon_{yy} - \varepsilon_y B_y/2B)/B = 0. \qquad \ldots(3.8)$$

This proves the theorem.

Theorem 4—The g_{ij} given by (1.8) is a solution of weakened field equation (1.5).

PROOF : The proof follows directly from (2.5*a*).

Theorem 5—A necessary and sufficient condition that g_{ij} given by (1.8) be a solution of weakened field equation (1.6) is

$$\varepsilon_x = 0, \quad \varepsilon_y = 0 \quad \text{and} \quad \bar{\epsilon} = 0. \qquad \ldots(3.9)$$

PROOF : When (2.1), (2.2) and (2.4) are used in (1.6), we find that it is satisfied for all values of i and j except for $i, j = 3$, 4 which yield the required result (3.9). Conversely, if (3.9) holds the field equation (1.6) is identically satisfied. Hence the theorem.

The space-time (1.8) is non-flat in general but g_{ij} is not the function of Z alone; the solutions obtained here can be called plane wave-like solutions following Takeno (1961). Choosing $A = B = 1$ in the solutions obtained above one can get easily the solutions of Lal and Pandey (1975).

REFERENCES

Buchdahl, H. A. (1962). On the gravitational field equations arising from the squar of the Gaussian curvature. *Nuovo Cim.*, **23**, 141–57.

Duplessis, J. C. (1965). Invariance properties of variational principles in general relativity. Ph.D. thesis, University of South Africa.

Kilmister, C. W., and Newmann, D. J. (1961). The use of algebraic structures in physics. *Proc. Camb. Phil. Soc.*, **57**, 851–64.

Lal, K. B., and Pandey, S. N. (1975). On plane wave-like solutions of weakened field equations in Peres space-time. *Tensor, N.S.*, **29**, 264–66.

Lal, K. B., and Singh, T. (1973). On cylindrical wave solutions of Kilmister and Newmann's weakened field equations in general relativity. *Tensor, N.S.*, **27**, 287–90.

Lovelock, D. (1967*a*). Weakened field equations in general relativity admitting an ' unphysical ' metric. *Commun. Math. Phys.*, **5**, 205–14.

——— (1967*b*). A spherically symmetric solution of Maxwell-Einstein equations. *Commun. Math. Phys.*, **5**, 257–61.

Pandey, S. N. (1975). Plane wave solutions of weakened field equations in general relativity. *Tensor, N.S.*, **29**, 297–98.

Pechlaner, E., and Sexl, R. (1966). On quadratic Lagrangians in general relativity. *Commun. Math. Phys.*, **2**, 165–75.

Rund, H. (1964). Variational problems in which the unknown functions are tensor components. *Second Colloquium on the Calculus of Variations, University of South Africa*, pp. 129–74.

——— (1966). Variational problems involving combined tensor field. *Abh. Math. Sem. Univ. Humburg*, **29**, 243–62.

——— (1967). Invariant theory of variational problems for geometric objects. *Tensor, N.S.*, **19**.

Swami, S. P. (1970). A note on weakened field equations $R_{ij;k} - R_{ik;j} = 0$. *Indian J. pure appl. Math.*, **1**, 485–91.

Takeno, H. (1961). The mathematical theory of plane gravitational waves in general relativity. *Sci. Rep. Res. Inst. Theor. Phys. Hir. Univ.*, **1**.

——— (1962). Gravitational null field admitting a parallel null vector field—The space-times H and P. *Tensor, N.S.*, **12**, 197–218.

Thompson, A. H. (1963). The investigation of a set of weakened field equations for general relativity. Contract AF 61 (052)—457 TN 10, Aerospace Research Laboratories, U.S.A.F.