VISCOUS AND THERMAL EFFECTS ON RANKINE-HUGONIOT JUMP CONDITIONS

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In this paper it is shown that while deriving the jump conditions across a shock wave the terms which are multiples of viscosity or conductivity with flow variable gradients cannot be neglected under the situations arising. For such situations, first order corrections to the Rankine–Hugoniot equations are introduced.

1. INTRODUCTION

The aim of this investigation is to show that there come situations with shocks where terms which are multiples of viscosity or conductivity with flow variable gradients cannot be neglected in deriving the jump conditions across a shock. For this purpose, we have selected the problem of shocks attached to a wedge in two-dimensional motion. Thomas (1948) has shown that the curvature of the shock becomes infinite just before the maximum deflection point. When the curvature is large, the gradients of flow variables behind the shock also become large and viscous and thermal effects can no more be neglected.

We can safely assume in this analysis that gradients outside the shock do not become infinite, otherwise the discontinuity will be smoothened out, that is, we take the region behind with high but not infinite gradients. The analysis shows that the jumps also depend upon a parameter like Reynolds number, $R = k\eta/m$ where k is the curvature of the shock, η the coefficient of viscosity, m the mass flux across the shock and the Prandtl number is taken to be unity.

2. DETERMINATION OF JUMP CONDITIONS

Following Prager (1961) we take the viscosity stress tensor as

$$T_{ij} = -p\delta_{ij} + \eta\,(v_{i,j} + v_{j,i} - \tfrac{2}{3}\,\delta_{ij}\,v_{k,k}). \qquad \ldots(1)$$

The flow variables on the two sides of the shock, for a fluid characterized by the constitutive equation (1), are connected by the following relations developed by Pant and Mishra (1963):

$$[v_i] = -\frac{\delta}{1+\delta}\,v_{1n}\,n_i + \frac{\eta}{m}\,[v_{1,k}]\,(n_k\,\delta_{1i} + n_1\delta_{ki} - 2n_1 n_k\,n_i), \qquad \ldots(2)$$

$$[p] = \frac{\delta}{1+\delta}\,\rho_{1}\,v_{1n}^{2} + \eta\,[v_{1,k}]\,(2n_1 n_k - \tfrac{2}{3}\,\delta_{1k}), \qquad \ldots(3)$$

where

$$\delta = \frac{[\rho]}{\rho_{1l}} \qquad \text{...(4)}$$

and

$$m\left[\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}\right] = \eta\,[(v_{i,k} + v_{k,i})\,v_k]\,n_i - \tfrac{2}{3}\eta\,[v_{n l}\,v_{1,1}] + \lambda\,[T_{,i}]\,n_i. \qquad \text{...(5)}$$

As coefficient of viscosity and thermal conductivity are small, we add to relations (2), (3) and (5) the values of gradients obtained by not taking viscous and thermal effects into account which is equivalent to neglecting second and higher powers in η and λ. Using the relations developed by Pant (1969) for fluids with no viscosity and heat transfer, we have

$$\begin{aligned} v_{1,k} = &- \frac{2(1+\delta_0)}{\delta_0}\,\frac{n_1 n_k}{(\gamma+1)\,v_{1nl}}\,k\left\{\frac{\delta_0}{(1+\delta_0)^2}\left(1 + \frac{\gamma+1}{2}\delta_0\right)v_{1nl}^2\right.\\ &\left.- \left(\frac{8}{\gamma+1} - \frac{\delta_0}{1+\delta_0}\right)v_{1tl}^2\right\} - k\,v_{1tl}\left(\delta_0 - \frac{4}{\gamma+1}\right)\lambda_1 n_k\\ &+ \left(\frac{4}{\gamma+1} - \frac{\delta_0}{1+\delta_0}\right)k\,v_{1tl}\,n_1\lambda_k + \frac{\delta_0}{1+\delta_0}\,k\;v_{1nl}\,\lambda_1\lambda_k, \end{aligned} \qquad \text{...(6)}$$

$$\begin{aligned} T_{,i}n_i = \frac{k}{\gamma R_g}(\gamma-1)&\left[\frac{2+(\gamma+1)\,\delta_0}{(\gamma+1)(1+\delta_0)^2}\,v_{1nl}^2 + v_{1tl}^2\left\{\left(\frac{4}{\gamma+1} + \frac{\delta_0^2}{1+\delta_0}\right)\right.\right.\\ &\left.\left.- \left(\frac{2}{(\gamma+1)\,\delta_0} + 1\right)\left(\frac{8}{\gamma+1} - \frac{\delta_0}{1+\delta_0}\right)\right\}\right] \end{aligned} \qquad \text{...(7)}$$

where R_g is the gas constant,

$$\delta_0 = \frac{2\,(M_{1nl}^2 - 1)}{2 + (\gamma-1)\,M_{1nl}^2}\,, \qquad \text{...(8)}$$

where δ_0 is the density strength of the shock without viscosity and heat conductivity.

From relation (6) we obtain

$$\begin{aligned} v_{1,k}\,(n_k\delta_{1i} + n_1\delta_{ki} - 2n_1n_kn_i) &= k\,v_{1tl}\left\{\frac{8}{\gamma+1} - \frac{\delta_0(2+\delta_0)}{1+\delta_0}\right\}\lambda_i\\ &= k\,v_{1tl}\,A_0\,\lambda_i \end{aligned} \qquad \text{...(9)}$$

and

$$\begin{aligned} v_{1,k}\left(2n_1n_k - \frac{2}{3}\delta_{1k}\right) = &- \frac{2}{3}\,\frac{4(1+\delta_0)}{\delta_0}\,\frac{kv_{1nl}}{(\gamma+1)}\left\{\frac{\delta_0}{(1+\delta_0)^2}\left(1 + \frac{\gamma+1}{2}\delta_0\right)\right.\\ &\left.+ \frac{\delta_0^2}{4(1+\delta_0)^2}(\gamma+1) - \left(\frac{8}{\gamma+1} - \frac{\delta_0}{1+\delta_0}\right)\cot^2\phi_{1l}\right\}\\ = &- \frac{2}{3}\,\frac{k}{m}\,B_0\,\rho_{1l}\,v_{1nl}^2 \end{aligned} \qquad \text{...(10)}$$

where $\cot \phi_{1l} = v_{1tl}/v_{1nl}$, ϕ_{1l} being the angle which the direction of flow in front makes with the tangent to the shock. The relations (2) and (3) using above expressions reduce to

$$[v_i] = -\frac{\delta}{1+\delta} v_{1nl} n_i + RA_0 v_{1tl} \lambda_i , \qquad ...(11)$$

$$[p] = \frac{\delta}{1+\delta} \rho_{1l} v_{1nl}^2 - \frac{2}{3} R B_0 \rho_{1l} v_{1nl}^2, \qquad ...(12)$$

where $R = k\eta/m$, A_0 and B_0 are known quantities as δ_0 is. In deriving above relations it has been assumed that $R^2 \ll 1$. The above relations now contain only one unknown quantity namely δ. To determine this we use energy balance (5). Under the approximation $R^2 \ll 1$ and the fact that η and λ are small, we have $\eta f(\delta) = \eta f(\delta_0)$. Substituting the values of different quantities in (5) and using relations (11) and (12), we obtain the following equation for δ:

$$\delta^2 \{(\gamma-1) v_{1nl}^2 + 2c_{1l}^2\} - 2\delta (v_{1nl}^2 - c_{1l}^2) + \tfrac{4}{3}(1+\delta_0) B_0 R v_{1nl}^2 + 2\frac{R}{Pr} C_0 = 0, \qquad ...(13)$$

where

$$C_0 = (\gamma-1)\Big[\frac{2+(\gamma+1)\delta_0}{\gamma+1} v_{1nl}^2 + \Big\{\frac{4(1+\delta_0)^2}{\gamma+1} + \delta_0^2 (1+\delta_0) - \Big(\frac{2}{(\gamma+1)\delta_0} + 1\Big)\Big(\frac{8}{\gamma+1} - \frac{\delta_0}{1+\delta_0}\Big)(1+\delta_0)^2\Big\} v_{1tl}^2\Big] \qquad (14)$$

and Pr is the Prandtl number.

3. The Case When Pr is Unity

We assume the effect of viscosity and heat transfer of the same order and hence take Prandtl number to be unity. Equation (13) gives

$$\delta = \delta_0 - RD_0 \qquad ...(15)$$

where

$$D_0 = \frac{2}{3(\gamma+1)} \frac{1+\delta_0}{\delta_0} \Big\{\Big(3(\gamma+1)\delta_0 + \frac{2(3\gamma+1)}{\gamma+1}\Big) + \frac{1+\delta_0}{\delta_0} \cot^2 \phi_{1l} \times \Big(3(\gamma-1)\delta_0^3 - 3\frac{\gamma-1}{\gamma+1}(3-\gamma)\delta_0^2 - \frac{2}{(\gamma+1)^2}(3\gamma^2+20\gamma+1)\delta_0 - \frac{16(3\gamma+1)}{(\gamma+1)^2}\Big)\Big\} . \qquad ...(16)$$

In the above we have considered the root which tends to δ_0 as R approaches zero and have used the approximation $R^2 \ll 1$. Equation (12) now reduces to

$$[p] = \frac{\delta_0}{1+\delta_0} \rho_{1l} v_{1nl}^2 - RE_0 \rho_{1l} v_{1nl}^2 , \qquad ...(17)$$

where

$$E_0 = \frac{2}{3(\gamma+1)}\,\frac{1}{\delta_0(1+\delta_0)}\left[\left(3(\gamma+1)\,\delta_0^2 + (3\gamma+7)\,\delta_0 + \frac{2(3\gamma+1)}{\gamma+1}\right) + \frac{1+\delta_0}{\delta_0}\cot^2\phi_{1l}\left\{3(\gamma-1)\,\delta_0^3 - \left(3\,\frac{\gamma-1}{\gamma+1}(3-\gamma) + \frac{32}{\gamma+1} - 4\right)\delta_0^2 - \left(\frac{2}{(\gamma+1)^2}(3\gamma^2+20\gamma+1) + \frac{32}{\gamma+1}\right)\delta_0 - \frac{16(3\gamma+1)}{(\gamma+1)^2}\right\}\right]. \quad \ldots(18)$$

Relation (15) shows that if D_0 is positive then the density behind the shock is less than the value calculated from the Rankine-Hugoniot relation and reverse is the case when D_0 is negative. Similar argument holds for pressure with respect to E_0.

In Fig. 1, using data from Thomas (1948), we have drawn the curve S which gives the position when the curvature of the shock becomes infinite. In the

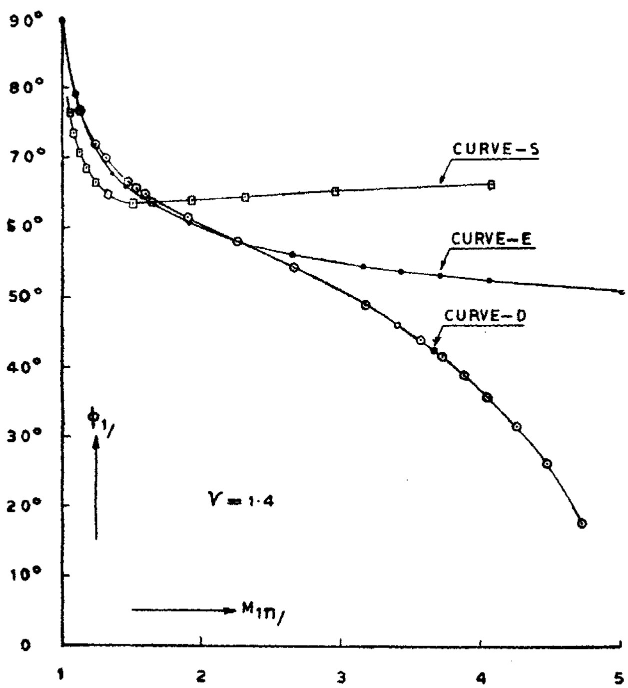

FIG. 1. Curve S, at which curvature of the shock becomes infinite; Curve D, above which D_0 is positive and below negative; Curve E, above which E_0 is positive and below negative.

near vicinity of this curve viscous and thermal effects become important, that is, R has a value such that the terms with it in relations (15) and (17) are not negligible. D_0 and E_0 are positive above curves D and E respectively and below negative.

As a first approximation following Pai (1959) the shock thickness L is given by

$$\eta\;\frac{v_{1nl} - v_{2nl}}{L} = p_{2l} - p_{1l}.$$

Using in above (11) and (18), we obtain

$$L = \frac{\eta}{\left(1 - \frac{1 + \delta_0}{\delta_0} RE_0\right) \rho_{1l} v_{1*l}} \qquad \ldots(19)$$

Figure 1 and relations (15), (17) and (19) show that viscosity and thermal conductivity do effect the flow pattern in shocks with large curvature.

REFERENCES

Pai, Shih-I. (1959). Introduction to the Theory of Compressible Flow. D. Van Nostrand Company, New York.

Pant, J. C. (1969). Some aspects of unsteady curved shock waves. *Int. J. engng. Sci.*, **7**, 235–45.

Pant, J. C., and Mishra, R. S. (1963). Shock waves of finite thickness in magnetogasdynamics. *Rend. Circ. Mat. Palermo*, Serie II, **12**, 1–23.

Prager, W. (1961). Introduction to Mechanics of Continua. Ginn and Company, New Yoık.

Thomas, T. Y. (1948). Calculation of curvature of attached shock waves. *J. Math. Phys.*, **27**, 279–97.