A NOTE ON NUMERICAL SOLUTION OF INTEGRAL EQUATIONS

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The integral equations are solved by Gaussian quadrature rule approximated by some finite sum rules. One may obtain such sum rules by conformal mapping $[0, \infty)$ into a finite interval $[-1, 1)$. This approach gives the good accuracy for the small number of mesh points.

§ 1. Here we have modified Osborn's (1967) techniques of solving the integral equations by Gaussian quadrature rule approximated by some finite sum rules. We define the general form of these rules as: Let the set

$$X(N) = \{x_i: x_i \in (a, b), \quad i \leqslant N\} \qquad ...(1.1)$$

be a discrete set of N points on the interval (a, b) of integration. Considering the integrand as a product of two factors $w(x)$ and $h(x)$, all the singularities and rapidly varying portions of the integrand are contained in $w(x)$, whereas $h(x)$ is a smooth, singularity-free function with a rapidly converging Taylor series. Then

$$\int_a^b w(x)\, h(x)\, dx = \sum_{i=1}^{N} h(x_i)\, w_i + \varepsilon_N(h), \qquad ...(1.2)$$

where ε_N is the error associated with this quadrature rule and factors w_i and $w(x)$ are usually called weights and weight-functions respectively. The weights $\{w_i\}$ are a set of N numbers and can be determined when ε_N is small. So the weights can be calculated when $\varepsilon_N(x^k) = 0$, for $k = 0, 1, \ldots, N-1$, i.e. we calculate

$$I_k = \int_a^b x^k\, w(x)\, dx \qquad ...(1.3)$$

either by some numerical scheme for each integer value of k or analytically and then solve the linear $N \times N$ matrix problem given by

$$I_k = x_1^k\, w_1 + x_2^k\, w_2 + \ldots + x_N^k\, w_N, \ (k = 0, 1, \ldots, N-1) \qquad ...(1.4)$$

This can be written in the matrix form

$$\begin{bmatrix} I_0 \\ I_1 \\ \cdot \\ \cdot \\ \cdot \\ I_{N-1} \end{bmatrix} = \begin{bmatrix} x_1^0 & x_2^0 & \cdots & x_N^0 \\ x_1^1 & x_2^1 & \cdots & x_N^1 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_N \end{bmatrix} \qquad \ldots(1.5)$$

It is, therefore, possible to calculate the weights w_i when I_k and x^k are known. We expand $h(x)$ in a Taylor series. Thus

$$h(x) = \sum_{n=0}^{N-1} a_n x^n + R_N(x) \qquad \ldots(1.6)$$

where $R_N(x)$ is the remainder term. By definition, the error term of weight is

$$\varepsilon_N(h) = \int_a^b R_N(x)\, w(x)\, dx. \qquad \ldots(1.7)$$

The first N terms in the power series for $h(x)$ are integrated exactly by the defined quadrature rule in eqn. (1.2). The quadrature rule is acceptable when $\varepsilon_N(h) < \delta$, where δ is some small positive number. We just increase N, for a given δ, until the above inequality is satisfied.

§ 2. We consider the Fredholm integral equation

$$u(x) = v(x) + \int_a^b K(x, y)\, u(y)\, dy,\ a \leqslant x \leqslant b, \qquad \ldots(2.1)$$

where the kernel $K(x, y)$ and $v(x)$ are known functions. We assume that a unique solution of $u(x)$ exists. The existence of a solution can usually be examined either by Fredholm theory or showing that the kernel $K(x, y)$ represents a compact operator. The first step in transforming eqn. (2.1) into a finite matrix form is to replace the integral by a sum. Thus we have

$$u(x) = v(x) + \sum_{i=1}^{N} K(x, y_i)\, u(y_i)\, w_i + \varepsilon_N\left(K(x, y_i)\, u(y_i)\right). \qquad \ldots(2.2)$$

This equation is valid for all $x \in (a, b)$. If we consider the x given by the N points subset $X(N)$ of (a, b), then eqn. (2.2) becomes N equations for N unknown $u(y_i)$. $\varepsilon_N\left(K(x, y_i)\, u(y_i)\right)$ may be dropped from the right-hand side of eqn. (2.2) for the accuracy of the quadrature rule when $\varepsilon_N\left(K(x, y_i)\, u(y_i)\right) < \delta$, where δ is sufficiently small. The resulting $N \times N$ matrix equation then becomes

$$\bar{u}(x_i) = v(x_i) + \sum_{j=1}^{N} K(x_i, y_j)\, \bar{u}(y_j)\, w_j. \qquad \ldots(2.3)$$

Equation (2.3) is completely different from eqn. (2.2) because of neglecting $\varepsilon_N\left(K(x, y_i)\, u(y_i)\right)$. The equation is also satisfied by $u(x_i)$

$$u(x_i) = v(x_i) + \sum_{j=1}^{N} K(x_i, y_j)\, u(y_j)\, w_j + \varepsilon_N\left(K(x_i, y_j)\, u(y_j)\right). \qquad \ldots(2.4)$$

Subtracting eqn. (2.3) from eqn. (2.4) and defining the error in $\bar{u}$ as

$$E(x_i) = u(x_i) - \bar{u}(x_i), \qquad \ldots(2.5)$$

then

$$E(x_i) = \varepsilon_N \left(K(x_i, y)\, u(y)\right) + \sum_{j=1}^{N} K(x_i, y_j)\, E(y_j)\, w_j. \qquad \ldots(2.6)$$

Thus the error is determined by

$$E(x_i) = \sum_{j} (\delta_{ij} - K_{ij} w_j)^{-1} \varepsilon_N \left(K(x_j, y)\, u(y)\right), \qquad \ldots(2.7)$$

where $(\delta_{ij} - K_{ij} w_j)^{-1}$ is the inverse of the matrix $\delta_{ij} - K(x_i, y_j) w_j$. Hence if $\left| \varepsilon_N \left(K(x_i, y)\, u(y)\right) \right| \ll u(x_i)$, we can expect the error in replacing $u(x_i)$ by $\bar{u}(x_i)$ to be small. If the $u(x_i)$ have been determined once, it is easy to find u for all x by using eqn. (2.2) and neglecting the error term. Caution is necessary when necessary $\varepsilon_N \left(K(x, y)\, u(y)\right)$. This technique of interpolating u is generally much more accurate than linear or Lagrangian methods.

§ 3. The quadrature rules like eqn. (1.2) are valid for infinite interval $[0, \infty)$. We obtain such rules by conformal mapping $[0, \infty)$ into a finite interval $[-1, 1)$ and the using Gaussian formulas for the finite interval.

Let us integrate $h(x)$ on $[0, \infty)$ according to the quadrature rule,

$$\int_0^{\infty} w(x)\, h(x)\, dx = \sum_{i=1}^{N} h(x_i)\, w_i. \qquad \ldots(3.1)$$

The problem is to determine w_i and x_i so that the approximation is the best we can obtain for N points.

Let us consider the map $x \to y \in [-1, 1)$ given by

$$y = \frac{x-1}{x+1}, \qquad x = \frac{1+y}{1-y}. \qquad \ldots(3.2)$$

The Jacobian is

$$\frac{dx}{dy} = \frac{2}{(1-y)^2} = \frac{(1+x)^2}{2} \geqslant 0. \qquad \ldots(3.3)$$

Thus the integral becomes

$$\int_0^{\infty} w(x)\, h(x)\, dx = \int_{-1}^{1} 2w\left(x(y)\right) h\left(x(y)\right) (1-y)^{-2}\, dy. \qquad \ldots(3.4)$$

Now the integral on the right is easily solvable by Gauss' method. This method is as

$$\int_{-1}^{1} u(y)\, dy = u \sum_{i=1}^{N} u(y_i)\, \bar{w}_i \qquad \ldots(3.5)$$

is exact for u in the class of polynomials of order $2N-1$ or less. Thus if the $2N$ components $\{y_1, y_2, \ldots, y_N; \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_N\}$ are chosen for $u \equiv (1, y^1, y^2, \ldots, y^{2N-1})$,

then eqn. (3.4) is exact. The values of abcissa y_i and weights $\bar{w}_i$ have been tabulated (Abramowitz and Stegun 1965) for many values of $N(\leqslant 100)$. By using the Gaussian abscissa and weights appropriate to eqn. (3.1), we obtain an integration rule for $h(x)$

$$\int_0^\infty w(x)\, h(x)\, dx = \sum_{i=1}^{N} h(x_i) \frac{(1+x_i)^2}{2} \bar{w}_i \qquad \ldots(3.6)$$

where the x_i are determined by eqn. (3.2) from y_i. The weights for quadrature rule are, obviously,

$$w_i = \frac{(1+x_i)^2}{2} \bar{w}_i. \qquad \ldots(3.7)$$

So eqn. (3.6) becomes

$$\int_0^\infty w(x)\, h(x)\, dx = \sum_{i=1}^{N} h(x_i)\, w_i. \qquad \ldots(3.8)$$

By this method we obtained the errors 0·26% and 0·008% for $N=5$ and 10 respectively. These estimates prove that our transformation technique is more accurate than that of Osborn (1967). This approach gives the good accuracy for $N \leqslant 10$.

REFERENCES

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