ON GENERALIZED K-RECURRENT FINSLER SPACES WITH A SYMMETRIC NON-ZERO RECURRENCE TENSOR FIELD

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We have defined a generalized K-recurrent Finsler space of the second order with the help of a symmetric non-zero recurrence tensor field a_{lm}. Certain theorems regarding it have been obtained in an affinely connected F_n.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n (Rund 1959), equipped with a positively homogeneous function $F(x, \dot{x})$ whose metric tensor $g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F(x, \dot{x})$, is symmetric in its lower indices and is positive homogeneous of degree zero in $\dot{x}^i$.

We have the following curvature tensors:

$$K^i_{jhk}(x, \dot{x}) = (\partial_k \Gamma^{*i}_{jh} - \dot{\partial}_l \Gamma^{*i}_{jh} \, \dot{\partial}_k G^l) - (\partial_h \Gamma^{*i}_{jk} - \dot{\partial}_l \Gamma^{*i}_{jk} \, \dot{\partial}_h G^l) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh} - \Gamma^{*i}_{mh} \Gamma^{*m}_{jk} \qquad \ldots(1.1a)$$

and

$$H^i_{hjk}(x, \dot{x}) = \partial_k G^i_{hj} - \partial_j G^i_{hk} + G^r_{hj} G^i_{rk} - G^r_{hk} G^i_{rj} + G^i_{rhk} \dot{\partial}_j G^r - G^i_{rhj} \dot{\partial}_k G^r \qquad \ldots(1.1b)$$

where

$$K^i_{hjk}(x, \dot{x}) \dot{x}^h = H^i_{hjk}(x, \dot{x}) \dot{x}^h = H^i_{hk}(x, \dot{x}) \qquad \ldots(1.2a)$$

$$G^i_{hk}(x, \dot{x}) = \dot{\partial}_h \dot{\partial}_k G^i(x, \dot{x}) \qquad \ldots(1.2b)$$

and

$$K_{jihk}(x, \dot{x}) \dot{x}^j = - K_{ijhk}(x, \dot{x}) \dot{x}^j. \qquad \ldots(1.2c)$$

$\partial_h = \dfrac{\partial}{\partial x^h}$ and $\dot{\partial}_h = \dfrac{\partial}{\partial \dot{x}^h}$

The covariant derivative of a vector $X^i(x, \dot{x})$ in the sense of Cartan is given by,

$$X^i_{|k} = \partial_k X^i - \dot{\partial}_h X^i \dot{\partial}_k G^h + X^k \Gamma^{*i}_{kj}. \qquad ...(1.3)$$

The Bianchi identity for K^i_{jhk} takes the following form

$$K^r_{ijk|h} + K^r_{ihj|k} + K^r_{ikh|j} + (\dot{\partial}_l \Gamma^{*r}_{ij} K^l_{skh} + \dot{\partial}_l \Gamma^{*r}_{ih} K^l_{sjk} + \dot{\partial}_l \Gamma^{*r}_{ik} K^l_{shj}) \dot{x}^s = 0. \qquad ...(1.4)$$

The following commutation formula for tensor field $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ is given by (Rund 1959)

$$T^{i_1 \ldots i_p}_{j_1 \ldots j_q|hk} - T^{i_1 \ldots i_p}_{j_1 \ldots j_q|kh}$$
$$= - \dot{\partial}_l T^{i_1 \ldots i_p}_{j_1 \ldots j_q} K^l_{rhk} \dot{x}^r + \sum_{\alpha=1}^{p} T^{i_1 \ldots i_{\alpha-1} r i_{\alpha+1} \ldots i_p}_{j_1 \ldots j_q} K^{i_\alpha}_{rhk}$$
$$- \sum_{\beta=1}^{q} T^{i_1 \ldots i_p}_{j_1 \ldots j_{\beta-1} s j_{\beta+1} \ldots j_q} K^s_{j_\beta hk} \qquad ...(1.5)$$

$$(\dot{\partial}_h T^{i_1 \ldots i_p}_{j_1 \ldots j_q})_{|m} - \dot{\partial}_h (T^{i_1 \ldots i_p}_{j_1 \ldots j_q|m})$$
$$= \dot{\partial}_l T^{i_1 \ldots i_p}_{j_1 \ldots j_q} C^l_{hm|r} \dot{x}^r - \sum_{\alpha=1}^{p} T^{i_1 \ldots i_{\alpha-1} r i_{\alpha+1} \ldots i_p}_{j_1 \ldots j_q} (\dot{\partial}_h \Gamma^{*i_\alpha}_{rm})$$
$$+ \sum_{\beta=1}^{q} T^{i_1 \ldots i_p}_{j_1 \ldots j_{\beta-1} s j_{\beta+1} \ldots j_q} (\dot{\partial}_h \Gamma^{*s}_{j_\beta m}). \qquad ...(1.6)$$

Definition 1 (Rund 1959)—Affinely connected space: The affinely connected Finsler spaces are characterized by the condition $C_{ijk|r} = 0$ where

$$C_{ijk} \overset{\text{def}}{=} \frac{1}{4} \frac{\partial^3 F^2}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k} \qquad \text{and} \qquad C^i_{jk|h} \dot{x}^h = \dot{\partial}_k \Gamma^{*i}_{jh} \dot{x}^h.$$

Definition 1.2 (Mishra and Pande 1968, Moór 1963)—Recurrent Finsler space of the first order: An n-dimensional Finsler space F_n is said to be recurrent if its curvature tensor $K^i_{jhk}(x, \dot{x})$ satisfies the relation

$$K^i_{jkh|l} = \beta_l K^i_{jkh}, \qquad ...(1.7)$$

where the non-zero vector β_l is the recurrence vector field.

2. Generalized K-Recurrent Finsler Spaces of the Second Order

Definition—An n-dimensional Finsler space F_n is said to be a generalized K-recurrent Finsler space of the second order if the curvature tensor $K^i_{jkh}(x, \dot{x})$, satisfies the relation

$$K^i_{jkh|lm} = \beta_m K^i_{jkh|l} + a_{lm} K^i_{jkh} \qquad ...(2.1)$$

where β_m and a_{lm} are non-zero and are called respectively associated vector and associated tensor (Ray 1972) of recurrence.

Theorem 2.1—For a generalized K-recurrent Finsler space in which the tensor of recurrence a_{lm} is symmetric and the curvature tensor field $K^i_{jkh}(x, \dot{x})$ is first order recurrent (Sinha and Singh 1970) with respect to the associated vector of recurrence, we have

$$K^i_{jkh|lm} - K^i_{jkh|ml} = 0. \qquad \ldots(2.2)$$

PROOF : Interchanging the indices l and m in (2.1) and subtracting the equation thus obtained from (2.1), we get

$$K^i_{jkh|lm} - K^i_{jkh|ml} = \beta_m K^i_{jkh|l} + a_{lm} K^i_{jkh} - \beta_l K^i_{jkh|m} - a_{ml} K^i_{jkh}. \qquad \ldots(2.3)$$

With the help of (1.7) and (2.3) we have

$$K^i_{jkh|lm} - K^i_{jkh|ml} = (a_{lm} - a_{ml}) K^i_{jkh}. \qquad \ldots(2.4)$$

From (2.4) we easily get the result in view of the symmetric property of a_{lm}.

Theorem 2.2—For a Finsler space with generalized K-recurrent curvature tensor field K^i_{jkh} in which the tensor of recurrence is symmetric, we have

$$\dot{\partial}_r K_{jk} H^r_{lm} = 2K_{p\,[j} K^p_{k]\,lm}. \qquad \ldots(2.5)$$

PROOF : Using the commutation formula (1.5) and (2.2), we get

$$- \dot{\partial}_r K^i_{jkh} K^r_{plm} \dot{x}^p + K^p_{jkh} K^i_{plm} - K^i_{pkh} K^p_{jlm} - K^i_{jph} K^p_{klm} - K^i_{jkp} K^p_{hlm} = 0 \qquad \ldots(2.6)$$

therefore we have

$$\dot{\partial}_r K^i_{jkh} K^r_{plm} \dot{x}^p - K^p_{jkh} K^i_{plm} + K^i_{pkh} K^p_{jlm} + K^i_{jph} K^p_{klm} + K^i_{jkp} K^p_{hlm} = 0. \qquad \ldots(2.7)$$

Contracting (2.7) with respect to the indices i and h we get

$$\dot{\partial}_r K_{jk} H^r_{lm} = - K_{pk} K^p_{jlm} - K_{jp} K^p_{klm} \qquad \ldots(2.8)$$

with the help of equations (1.2a), (1.2c) and (2.8) we get the required result.

Theorem 2.3—Finsler space with the generalized K-recurrent curvature tensor field K^i_{jkh}, having a symmetric recurrence tensor a_{lm}, we have

$$a_{hm} K^r_{ijk} + a_{km} K^r_{ihj} + a_{jm} K^r_{ikh} + \{(\dot{\partial}_l \Gamma^{*r}_{ij})_{|m} K^l_{skh} + (\dot{\partial}_l \Gamma^{*r}_{ih})_{|m} K^l_{sjk} + (\dot{\partial}_l \Gamma^{*r}_{ik})_{|m} K^l_{shj}\} \dot{x}^s = 0. \qquad \ldots(2.9)$$

PROOF : Differentiating (1.4) covariantly with respect to $\dot{x}^m$ in the sense of Cartan and using (1.7), we get

$$a_{hm} K^r_{ijk} + a_{km} K^r_{ihj} + a_{jm} K^r_{ikh} + \beta_m (K^r_{ihj|k} + K^r_{ijk|h} + K^r_{ikh|j}) + (\dot{\partial}_l \Gamma^{*r}_{ij})_{|m} K^l_{skh} + (\dot{\partial}_l \Gamma^{*r}_{ih})_{|m} K^l_{sjk} + (\dot{\partial}_l \Gamma^{*r}_{ik})_{|m} K^l_{shj} + \beta_m \{(\dot{\partial}_l \Gamma^{*r}_{ij}) K^l_{skh} + (\dot{\partial}_l \Gamma^{*r}_{ih}) K^l_{sjk} + (\dot{\partial}_l \Gamma^{*r}_{ik}) K^l_{shj}\} \dot{x}^s = 0 \qquad \ldots(2.10)$$

with the help of (1.4) and (2.10), we obtain

$$a_{hm}K^r{}_{ijk} + a_{km}K^r{}_{ihj} + a_{jm}K^r{}_{ikh} + \beta_m \{(\dot{\partial}_l\Gamma^{*r}_{ij})\, K^l{}_{skh} + (\dot{\partial}_l\Gamma^{*r}_{ih})\, K^l{}_{sjk}$$
$$+ (\dot{\partial}_l\Gamma^{*r}_{ik})\, K^l{}_{shj}\}\, \dot{x}^s + \dot{x}^s \{(\dot{\partial}_l\Gamma^{*r}_{ij})_{|m}K^l{}_{skh} + (\dot{\partial}_l\Gamma^{*r}_{ih})_{|m}K^l{}_{sjk}$$
$$+ (\dot{\partial}_l\Gamma^{*r}_{ik})_{|m}K^l{}_{shj}\} - \beta_m\, \dot{x}^s \{(\dot{\partial}_l\Gamma^{*r}_{ij})\, K^l{}_{skh} + (\dot{\partial}_l\Gamma^{*r}_{ih})\, K^l{}_{sjk}$$
$$+ (\dot{\partial}_l\Gamma^{*r}_{ik})\, K^l{}_{shj}\} = 0 \qquad \ldots(2.11)$$

which yields the Theorem 2.3.

Theorem 2.4—In an affinely connected Finsler space (Pande and Khan 1973) with a generalized K-recurrent curvature tensor field, the relation

$$a_{hm}K^r{}_{ijk} + a_{km}K^r{}_{ihj} + a_{jm}K^r{}_{ikh} = 0 \qquad \ldots(2.12)$$

holds.

PROOF: The proof is obvious from (2.11) and the definition (1.1).

Theorem 2.5—In a Finsler space with generalized K-recurrent curvature tensor field $K^r{}_{ijk}$ and having a symmetric recurrent tensor a_{lm}, we have

$$K^r{}_{ijk|hlm} - K^r{}_{ijk|lhm} = (a_{hm}\beta_l - a_{lm}\beta_h + \beta_h\beta_{l|m} - \beta_l\beta_{h|m})\, K^r{}_{ijk}. \qquad \ldots(2.13)$$

PROOF: From equation (2.1) we get

$$K^r{}_{ijk|hl} = \beta_l K^r{}_{ijk|h} + a_{hl}K^r{}_{ijk}. \qquad \ldots(2.14)$$

Differentiating (2.14) covariantly with respect to x^m we have

$$K^r{}_{ijk|hlm} = K^r{}_{ijk|hm}\beta_l + K^r{}_{ijk|h}\beta_{l|m} + a_{hl|m}K^r{}_{ijk} + a_{hl}K^r{}_{ijk|m}. \qquad \ldots(2.15)$$

Interchanging the indices h and l in (2.15), subtracting it from the above relation (2.15) and applying relations (2.1) and (1.7) we obtain

$$K^r{}_{ijk|hlm} - K^r{}_{ijk|lhm} = K^r{}_{ijk} \{(a_{hm}\beta_l - a_{lm}\beta_h) + (\beta_h\beta_{l|m} - \beta_l\beta_{h|m})$$
$$+ (a_{hl|m} - a_{lh|m}) + \beta_m\,(a_{hl} - a_{lh})\} \qquad \ldots(2.16)$$

using the symmetric property of a_{lm}, we get the result from (2.16).

Theorem 2.6—For an affinely connected Finsler space with the generalized K-recurrent curvature tensor field $K^i{}_{jkh}$ we have

$$2\beta_m \{K^n{}_{ijk}K^r{}_{nhl} - K^n{}_{ihl}K^r{}_{njk} - K^n{}_{jhl}K^r{}_{inh} - K^r{}_{ijn}K^n{}_{khl}\}$$
$$- K^n{}_{shl}\{\dot{\partial}_n\beta_m + 2\beta_m\dot{\partial}_n K^r{}_{ijk}\}\, \dot{x}^s$$
$$= K^r{}_{ijk}\{(a_{hm} - \beta_{h|m})\,\beta_l + (\beta_{l|m} - a_{lm})\,\beta_h\}. \qquad \ldots\mathbf{(2.17)}$$

PROOF: From the Theorem 2.5, we have

$$K^r{}_{ijk|hlm} - K^r{}_{ijk|lhm} = (a_{hm}\beta_l - a_{lm}\beta_h + \beta_h\beta_{l|m} - \beta_l\beta_{h|m})\, K^r{}_{ijk}. \qquad \ldots(2.18)$$

Using the commutation formula (1.5) we have

$$K^r_{ijk|hl} - K^r_{ijk|lh}$$
$$= -\dot{\partial}_n K^r_{ijk} K^n_{shl}\dot{x}^s + K^n_{ijk}K^r_{nhl} - K^r_{njk}K^n_{ihl} - K^r_{ink}K^n_{jhl} - K^r_{ijn}K^n_{khl}. \qquad ...(2.19)$$

Taking the covariant derivative of (2.19) with respect to $\dot{x}^m$ we obtain

$$K^r_{ijk|hlm} - K^r_{ijk|lhm}$$
$$= -(\dot{\partial}_n K^r_{ijk})_{|m} K^n_{shl}\dot{x}^s - (\dot{\partial}_n K^r_{ijk}) K^n_{shl|m}\dot{x}^s$$
$$+ K^n_{ijk|m}K^r_{nhl} + K^n_{ijk}K^r_{nhl|m} - K^r_{njk|m}K^n_{ihl} - K^r_{njk}K^n_{ihl|m}$$
$$- K^r_{ink|m}K^n_{jhl} - K^r_{ink}K^n_{jhl|m} - K^r_{ijn|m}K^n_{khl} - K^r_{ijn}K^n_{khl|m} \qquad ...(2.20)$$

because the covariant derivative of $\dot{x}^i$ in the sense of Cartan is zero. With the help of (1.6), we get

$$(\dot{\partial}_n K^r_{ijk})_{|m} = \dot{\partial}_n (K^r_{ijk|m}) + \dot{\partial}_p (K^r_{ijk}) C^p_{mn|q}\dot{x}^q - K^p_{ijk}(\dot{\partial}_n \Gamma^{*r}_{pm})$$
$$+ K^r_{pjk}(\dot{\partial}_n \Gamma^{*p}_{im}) + K^r_{ipk}(\dot{\partial}_n \Gamma^{*p}_{jm}) + K^r_{ijp}(\dot{\partial}_n \Gamma^{*p}_{km}). \qquad ...(2.21)$$

From (2.20) and (2.21), we obtain

$$K^r_{ijk|hlm} - K^r_{ijk|lhm}$$
$$= -\{\dot{\partial}_n (K^r_{ijk|m}) + (\dot{\partial}_p K^r_{ijk}) C^p_{mn|q}\dot{x}^q - K^p_{ijk}(\dot{\partial}_n \Gamma^{*r}_{pm})$$
$$+ K^r_{pjk}(\dot{\partial}_n \Gamma^{*p}_{im}) + K^r_{ipk}(\dot{\partial}_n \Gamma^{*p}_{jm}) + K^r_{ijp}(\dot{\partial}_n \Gamma^{*p}_{km})\} K^n_{shl}\dot{x}^s$$
$$- (\dot{\partial}_n K^r_{ijk}) K^n_{shl|m}\dot{x}^s + K^n_{ijk|m}K^r_{nhl} + K^n_{ijk}K^r_{nhl|m}$$
$$- K^r_{njk|m}K^n_{ihl} - K^r_{njk}K^n_{ihl|m} - K^r_{ink|m}K^n_{jhl}$$
$$- K^r_{ink}K^n_{jhl|m} - K^r_{ijn|m}K^n_{khl} - K^r_{ijn}K^n_{khl|m}. \qquad ...(2.22)$$

Using (1.7) and (2.18) in (2.22) we have

$$-(\dot{\partial}_n K^r_{ijk})\beta_m K^n_{shl}\dot{x}^s - (\dot{\partial}_n \beta_m) K^r_{ijk}K^n_{shl}\dot{x}^s - (\dot{\partial}_p K^r_{ijk})(\dot{\partial}_n \Gamma^{*p}_{mq})\dot{x}^s\dot{x}^q K^n_{shl}$$
$$+ K^p_{ijk}K^n_{shl}(\dot{\partial}_n \Gamma^{*r}_{pm})\dot{x}^s - K^r_{pjk}K^n_{shl}(\dot{\partial}_n \Gamma^{*p}_{im})\dot{x}^s - K^r_{ipk}K^n_{shl}(\dot{\partial}_n \Gamma^{*p}_{jm})\dot{x}^s$$
$$- K^r_{ijp}K^n_{shl}(\dot{\partial}_n \Gamma^{*p}_{km})\dot{x}^s - K^n_{shl}\beta_m(\dot{\partial}_n K^r_{ijk})\dot{x}^s + \beta_m K^n_{ijk}K^r_{nhl}$$
$$+ \beta_m K^n_{ijk}K^r_{nhl} - \beta_m K^r_{njk}K^n_{ihl} - \beta_m K^r_{ink}K^n_{jhl} - \beta_m K^r_{njk}K^n_{ihl}$$
$$- \beta_m K^r_{ink}K^n_{jhl} - \beta_m K^r_{ijn}K^n_{khl} - \beta_m K^r_{ijn}K^n_{khl}$$
$$= K^r_{ijk}\{(a_{hm} - \beta_{h|m})\beta_l + (\beta_{l|m} - a_{lm})\beta_h\}. \qquad ...(2.23)$$

The above equation yields the required result in view of the condition of Definition 1.1.

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