ON INFINITESIMAL CONFORMAL TRANSFORMATION IN A FINSLER SPACE

by H. D. PANDE *and* H. S. SHUKLA, *Department of Mathematics, University of Gorakhpur, Gorakhpur* 273001

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Sinha (1971) has studied infinitesimal conformal transformation in a Finsler space where he has derived the Lie-derivatives of the curvature tensors K^i_{hjk} and H^i_{hjk}. In the present paper, we have derived the Lie-derivatives of Cartan's first and second curvature tensors. Infinitesimal conformal transformations which leave invariant the curvature tensor and its covariant derivative have also been studied.

1. INTRODUCTION

We consider an n-dimensional Finsler space with its metric function $F(x, \dot{x})$ which is positively homogeneous of first degree in its directional arguments. The metric tensor $g_{ij}(x, \dot{x})$ is defined by

$$g_{ij} = \tfrac{1}{2}\, \dot{\partial}_i \dot{\partial}_j F^2, \qquad \left(\dot{\partial}_k \equiv \frac{\partial}{\partial \dot{x}^k}\right). \qquad \text{...(1.1)}$$

The tensor $C_{ijk}(x, \dot{x})$ defined by

$$C_{ijk} = \tfrac{1}{2}\, \dot{\partial}_k\, g_{ij} \qquad \text{...(1.2)}$$

is symmetric in all its three indices. The raising of indices is according to

$$C^i_{\ jk} = g^{ia}\, C_{jak} \qquad \text{...(1.3)}$$

where

$$g^{ij}\, g_{jk} = \delta^i_{\ k}. \qquad \text{...(1.4)}$$

The covariant derivative of a tensor $T^i_{\ j}(x, \dot{x})$ in the sense of Cartan and Berwald is given by

$$T^i_{\ j|k} = \partial_k T^i_j - \dot{\partial}_a T^i_j\, \Gamma^{*a}_{bk}\, \dot{x}^b + T^a_j\, \Gamma^{*i}_{ak} - T^i_a\, \Gamma^{*a}_{jk}, \left(\partial_k \equiv \frac{\partial}{\partial x^k}\right) \qquad \text{...(1.5)}$$

and

$$T^i_{\ j(k)} = \partial_k T^i_j - \dot{\partial}_a T^i_j\, G^a_{\ bk}\, \dot{x}^b + T^a_j\, G^i_{\ ak} - T^i_a\, G^a_{\ jk}, \qquad \text{...(1.6)}$$

respectively, where $\Gamma^{*i}_{jk}(x, \dot{x})$ and $G^i_{\ jk}(x, \dot{x})$ are symmetric connection coefficients of Cartan and Berwald respectively.

The tensor $\dot{\partial}_l \Gamma^{*i}_{jk}$ is given by

$$\dot{\partial}_l \Gamma^{*i}_{jk} = C^i{}_{jl|k} + C^i{}_{kl|j} - C^b{}_{jl|a} g_{bk} g^{ai} - C^i{}_{ja} C^a{}_{kl|b} \dot{x}^b - C^i{}_{ka} C^a{}_{jl|b} \dot{x}^b + C^a{}_{jk} C^i{}_{al|b} \dot{x}^b. \quad ...(1.7)$$

The Cartan's first and second curvature tensors are given by

$$S^i{}_{jkh} = A^i{}_{ka} A^a{}_{jh} - A^i{}_{ah} A^a{}_{jk} \quad ...(1.8)$$

and

$$P^i{}_{jkh} = A^i{}_{kh|j} - g^{im} A_{jkh|m} - C^i{}_{km} A^m{}_{jh|a} \dot{x}^a + C^m{}_{jk} A^i{}_{mh|a} \dot{x}^a \quad ...(1.9)$$

respectively where $A^i{}_{jk}(x, \dot{x})$ is defined by

$$A^i{}_{jk} = F C^i{}_{jk}. \quad ...(1.10)$$

Let us consider an infinitesimal transformation

$$\bar{x}^i = x^i + v^i(x)\, d\tau, \quad ...(1.11)$$

where $v^i(x)$ is a contravariant vector field defined over the domain of the space under consideration and $d\tau$ is an infinitesimal constant. Then, denoting by $\mathcal{L}$ the Lie-derivative with respect to the vector field v^i, we have (Yano 1957, Hiramatu 1954):

$$\dot{\partial}_l (\mathcal{L} T^i{}_{jk})\, \mathcal{L}(\dot{\partial}_l T^i{}_{jk}) = 0 \quad ...(1.12)$$

$$(\mathcal{L} T^i{}_{jk})_{|l} - \mathcal{L}(T^i{}_{jk|l}) = - T^a{}_{jk} \mathcal{L}\Gamma^{*i}_{al} + T^i{}_{ak} \mathcal{L}\Gamma^{*a}_{jl} + T^i{}_{ja} \mathcal{L}\Gamma^{*a}_{kl} + \dot{\partial}_a T^i{}_{jk} \mathcal{L}\Gamma^{*a}_{bl} \dot{x}^b, \quad ...(1.13)$$

$$\dot{\partial}_l (\mathcal{L}\Gamma^{*i}_{jk}) - \mathcal{L}(\dot{\partial}_l \Gamma^{*i}_{jk}) = 0 \quad ...(1.14)$$

$$(\mathcal{L}\Gamma^{*i}_{jk})_{|l} - (\mathcal{L}\Gamma^{*i}_{jl})_{|k} = \mathcal{L}K^i{}_{jkl} + \dot{\partial}_a \Gamma^{*i}_{jk}\, \mathcal{L}\, \Gamma^{*a}_{bl} \dot{x}^b - \dot{\partial}_a \Gamma^{*i}_{jl} \mathcal{L}\Gamma^{*a}_{bk} \dot{x}^b \quad ...(1.15)$$

where

$$K^i{}_{jkl} = (\partial_l \Gamma^{*i}_{jk} - \dot{\partial}_a \Gamma^{*i}_{jk} \Gamma^{*a}_{bl} \dot{x}^b) - (\partial_k \Gamma^{*i}_{jl} - \dot{\partial}_a \Gamma^{*i}_{jl} \Gamma^{*a}_{bk} \dot{x}^b) + \Gamma^{*a}_{jk} \Gamma^{*i}_{al} - \Gamma^{*a}_{jl} \Gamma^{*i}_{ak}$$

and

$$(\mathcal{L} G^i{}_{jh})_{(k)} - (\mathcal{L} G^i{}_{kh})_{(j)} = \mathcal{L} H^i{}_{hjk} + \dot{\partial}_a G^i{}_{jh} \mathcal{L} G^a{}_{bk} \dot{x}^b - \dot{\partial}_a G^i{}_{kh} \mathcal{L} G^a{}_{bj} \dot{x}^b \quad ...(1.16)$$

where

$$H^i_{hjk} = \partial_k G^i_{hj} - \partial_j G^i_{hk} + G^a_{hj} G^i_{ak} - G^a_{hk} G^i_{aj} + G^i_{ahk} \dot{\partial}_j G^a - G^i_{ahj} \dot{\partial}_k G^a.$$

2. Infinitesimal Conformal Transformation

Let us consider an infinitesimal transformation (1.11) which is defined as one by which the magnitudes of vectors defined in the same tangent space are proportional, and furthermore, that the angle between two directions in the same tangent space, is the same with respect to both metrices. We call such a transformation an infinitesimal conformal transformation. Consequently, if we use the Lie-derivatives, this transformation may be characterized by

$$\mathcal{L} g_{ij} = 2\sigma(x) g_{ij} \qquad \ldots(2.1)$$

where σ is independent of $\dot{x}^i$. If σ is constant, the transformation becomes homothetic and if σ is zero, the transformation becomes a motion.

For an infinitesimal conformal transformation, we have the following results (Sinha 1971):

$$\mathcal{L}\, \Gamma^{*i}_{jk} = M^{ih}_{jk} \sigma_h \qquad \ldots(2.2)$$

where

$$M^{ih}_{jk} \overset{\text{def}}{=} \delta_j^i \delta_k^h + \delta_k^i \delta_j^h - g^{ih} g_{jk} + \dot{\partial}_j B^{ah} C^i_{ka} + \dot{\partial}_k B^{ah} C^i_{ja} - g^{im} C_{jka} \dot{\partial}_m B^{ah},$$

$$B^{ik} \overset{\text{def}}{=} \tfrac{1}{2} F^2 g^{ik} - \dot{x}^i \dot{x}^k, \qquad \sigma_k \overset{\text{def}}{=} \partial_k \sigma,$$

and

$$\mathcal{L}\, G^i_{jk} = -\dot{\partial}_j \dot{\partial}_k B^{ih} \sigma_h. \qquad \ldots(2.3)$$

Theorem 2.1—The infinitesimal conformal transformation of Cartan's first curvature tensor is given by

$$\mathcal{L}\, S^i_{jkh} = 2\sigma S^i_{jkh}. \qquad \ldots(2.4)$$

Proof : From (1.4) and (2.1), we have

$$\mathcal{L}\, g^{ij} = -2\sigma g^{ij}. \qquad \ldots(2.5)$$

Applying (1.12) to g_{ij} and using (1.2) we have

$$\mathcal{L}\, C_{ijk} = 2\sigma C_{ijk} \qquad \ldots(2.6)$$

from which

$$\mathcal{L}\, C^i_{jk} = 0 \qquad \ldots(2.7)$$

by virtue of (1.3) and (2.5).

From (1.1), the homogeneity property of $F(x, \dot{x})$ and (2.1), we have

$$\mathcal{L}\, F = \sigma F. \qquad \ldots(2.8)$$

In view of (1.10), (2.7) and (2.8), we have

$$\mathcal{L}\, A^i_{jk} = \sigma A^i_{jk}. \qquad \ldots(2.9)$$

Taking the Lie-derivative of (1.8) and using (2.9), we have (2.4).

Theorem 2.2—The infinitesimal conformal transformation of Cartan's second curvature tensor is given by

$$\begin{aligned}
\mathcal{L}\, P^i_{jkh} = {} & \sigma \,(A^i_{kh|j} - g^{im} A_{jkh|m} - C^i_{km} A^m{}_{h|p}\, \dot{x}^p \\
& + C^m_{jk} A^i_{mh|p}\, \dot{x}^p) + \sigma_j A^i_{kh} - \sigma_m g^{im} A_{jkh} \\
& - \sigma_p (C^i_{km} A^m_{jh} - C^m_{jk} A^i_{mh})\, \dot{x}^p + \sigma_p [A^a_{kh} M^{ip}_{aj} \\
& - A^i_{ah} M^{ap}_{kj} - A^i_{ka} M^{ap}_{hj} - \dot{\partial}_a A^i_{kh} M^{ap}_{bj}\, \dot{x}^b \\
& - g^{im} \{g_{ka} (A^b_{jh} M^{ap}_{bm} - \dot{\partial}_b A^a_{jh} M^{bp}_{cm}\, \dot{x}^c) - A_{bkh} M^{bp}_{jm} \\
& - A_{jkb} M^{bp}_{hm}\} - C^i_{km} (A^a_{jh} M^{mp}_{aq} - A^m_{ah} M^{ap}_{jq} - A^m_{ja} M^{ap}_{hq} \\
& - \dot{\partial}_a A^m_{jh} M^{ap}_{bq}\, \dot{x}^b)\, \dot{x}^q + C^m_{jk} (A^a_{mh} M^{ip}_{aq} - A^i_{ah} M^{p}_{mq} \\
& - A^i_{ma} M^{ap}_{hq} - \dot{\partial}_a A^i_{mh} M^{ap}_{bq}\, \dot{x}^b)\, \dot{x}^q].
\end{aligned} \qquad \ldots(2.10)$$

Proof: Applying the formula (1.13) to A^i_{jk} and using (2.2) and (2.9), we have

$$\begin{aligned}
\mathcal{L}\,(A^i_{jk|l}) = {} & \sigma A^i_{jk|l} + \sigma_l A^i_{jk} + A^a_{jk} M^{ip}_{al}\, \sigma_p - A^i_{ak} M^{ap}_{jl}\, \sigma_p \\
& - A^i_{ja} M^{ap}_{kl}\, \sigma_p - \dot{\partial}_a A^i_{jk} M^{ap}_{bl}\, \sigma_p\, \dot{x}^b.
\end{aligned} \qquad \ldots(2.11)$$

Using $g_{ij|k} = 0$ and (2.1), we have from (2.11)

$$\begin{aligned}
\mathcal{L}\,(A_{ijk|l}) = {} & 3\sigma A_{ijk|l} + \sigma_l A_{ijk} + \{g_{ja} (A^b_{ik} M^{ap}_{bl} \\
& - \dot{\partial}_b A^a_{ik} M^{bp}_{cl}\, \dot{x}^c) - A_{bjk} M^{bp}_{il} - A_{ijb} M^{bp}_{kl}\}\, \sigma_p.
\end{aligned} \qquad \ldots(2.12)$$

Taking the Lie-derivative of (1.9) and using (2.5), (2.7), (2.11), (2.12) and the fact that $\mathcal{L}\, \dot{x}^a = 0$, we have (2.10).

Theorem 2.3—For an infinitesimal conformal transformation characterized by (2.1), we have

$$(g_{ak} M^{ah}_{jl} + g_{ja} M^{ah}_{kl} + 2C_{jka} M^{ah}_{bl} \dot{x}^b)\, \sigma_h - 2\sigma_l g_{jk} = 0. \qquad \ldots(2.13)$$

PROOF: Applying the formula (1.13) to g_{jk} and using (2.1), (2.2) and the fact that $g_{jk|l} = 0$, we have (2.13).

Theorem 2.4—For an infinitesimal conformal transformation, characterized by (2.1), we have

$$\begin{aligned}
[C^a{}_{jl} M^{ih}_{ak} &- C^i{}_{al} M^{ah}_{jk} - C^i{}_{ja} M^{ah}_{lk} - \dot{\partial}_a C^i{}_{jl} M^{ah}_{bk} \dot{x}^b \\
&+ C^a{}_{kl} M^{ih}_{aj} - C^i{}_{al} M^{ah}_{kj} - C^i{}_{ka} M^{ah}_{lj} - \dot{\partial}_a C^i{}_{kl} M^{ah}_{bj} \dot{x}^b \\
&- g_{bk} g^{ai} \{C^p{}_{jl} M^{bh}_{pa} - C^b{}_{pl} M^{ph}_{ja} - C^b{}_{jp} M^{ph}_{la} - \dot{\partial}_p C^b{}_{jl} M^{ph}_{qa} \dot{x}^q\} \\
&- C^i{}_{ja} \{C^p{}_{kl} M^{ch}_{pb} - C^a{}_{pl} M^{ph}_{kb} - C^a{}_{jp} M^{ph}_{lb} - \dot{\partial}_p C^i{}_{ja} M^{ph}_{qb} \dot{x}^q\} \dot{x}^b \\
&- C^i{}_{ka} \{C^p{}_{jl} M^{ah}_{pb} - C^a{}_{pl} M^{ph}_{jb} - C^a{}_{jp} M^{ph}_{lb} - \dot{\partial}_p C^i{}_{ka} M^{ph}_{qb} \dot{x}^q\} \dot{x}^b \\
&+ C^a{}_{jk} \{C^p{}_{al} M^{ih}_{pb} - C^i{}_{pl} M^{ph}_{ab} - C^i{}_{ap} M^{ph}_{lb} \\
&- \dot{\partial}_p C^i{}_{al} M^{ph}_{qb} \dot{x}^q\} \dot{x}^b - \dot{\partial}_l M^{ih}_{jk}]\, \sigma_h = 0 \qquad (2.14)
\end{aligned}$$

PROOF: From (1.14) and (2.2), we have

$$\mathcal{L}(\dot{\partial}_l \Gamma^{*i}_{jk}) = \dot{\partial}_l M^{ih}_{jk} \sigma_h. \qquad \ldots(2.15)$$

Taking the Lie-derivative of (1.7) and using (2.1), (2.5) and (2.7) we have

$$\begin{aligned}
\mathcal{L}(\dot{\partial}_l \Gamma^{*i}_{jk}) &= \mathcal{L}(C^i{}_{jl|k}) + \mathcal{L}(C^i{}_{kl|j}) - g_{bk} g^{ai} \mathcal{L}(C^b{}_{jl|a}) \\
&- C^i{}_{ja} \mathcal{L}(C^a{}_{kl|b}) \dot{x}^b - C^i{}_{ka} \mathcal{L}(C^a{}_{jl|b}) \dot{x}^b + C^a{}_{jk} \mathcal{L}(C^i{}_{al|b}) \dot{x}^b \qquad \ldots(2.16)
\end{aligned}$$

Applying (1.13) to $C^i{}_{jl}$ and noting (2.7) and (2.2), we have

$$\mathcal{L}(C^i{}_{jl|k}) = (C^a{}_{jl} M^{ih}_{ak} - C^i{}_{al} M^{ah}_{jk} - C^i{}_{ja} M^{ah}_{lk} - \dot{\partial}_a C^i{}_{jl} M^{ah}_{bk} \dot{x}^b)\, \sigma_h. \ldots(2.17)$$

Substituting (2.17) into (2.16) and using (2.15), we have (2.14).

3. Infinitesimal Conformal Transformations which Leave Invariant the Curvature Tensor and Its Covariant Derivative

Theorem 3.1—If the infinitesimal conformal transformation characterized by (2.1) leaves invariant the curvature tensor $K^i{}_{jkl}$ and also its covariant derivative, then the relation

$$(K^{a}{}_{jkh}\, M^{ip}_{al} - K^{i}{}_{akh}\, M^{ap}_{jl} - K^{i}{}_{jah}\, M^{ap}_{kl} - K^{i}{}_{jka}\, M^{ap}_{hl}$$

$$- \dot{\partial}_a K^{i}{}_{jkh}\, M^{ap}_{bl}\, \dot{x}^b)\, \sigma_p = 0 \qquad ...(3.1)$$

holds.

PROOF : Applying the formula (1.13) to $K^i{}_{jkh}$ and using (2.2), we have

$$\mathcal{L}(K^{i}{}_{jkh|l}) = (\mathcal{L}K^{i}{}_{jkh})_{|l} + (K^{a}{}_{jkh}\, M^{ip}_{al} - K^{i}{}_{akh}\, M^{ap}_{jl}$$

$$- K^{i}{}_{jah}\, M^{ap}_{kl} - K^{i}{}_{jka}\, M^{ap}_{hl} - \dot{\partial}_a K^{i}{}_{jkh}\, M^{ap}_{bl}\, \dot{x}^b)\, \sigma_p. \qquad ...(3.2)$$

If the infinitesimal conformal transformation leaves invariant the curvature tensor $K^i{}_{jkh}$ and its covariant derivative, then

$$\mathcal{L}\, K^{i}{}_{jkh} = 0 \qquad ...(3.3)$$

and

$$\mathcal{L}(K^{i}{}_{jkh|l}) = 0. \qquad ...(3.4)$$

Substituting (3.3) and (3.4) in (3.2), we obtain (3.1)

Theorem 3.2—If a K-symmetric Finsler space admits an infinitesimal conformal transformation under which the curvature tensor $K^i{}_{jkh}$ is invariant, then the relation (3.1) holds.

PROOF : If the Finsler space is K-symmetric, i.e. if $K^i{}_{jkh|l} = 0$, then the condition (3.4) is always satisfied. The theorem thus follows from (3.2), (3.3) and (3.4).

Theorem 3.3—If a Finsler space admits an infinitesimal conformal transformation which leaves invariant the curvature tensor $H^i{}_{jkl}$ and its covariant derivative, then we have

$$(\dot{\partial}_p B^{iq} H^{p}{}_{jkl} - \dot{\partial}_j B^{pq} H^{i}{}_{pkl} - \dot{\partial}_k B^{pq} H^{i}{}_{jpl} - \dot{\partial}_l B^{pq} H^{i}{}_{jkp}$$

$$- F^2 g^{pq}\, \dot{\partial}_p H^{i}{}_{jkl})\, \sigma_q = 0. \qquad ...(3.5)$$

PROOF : Applying the formula

$$\mathcal{L}(T^{i}{}_{j(a)}) - (\mathcal{L}\, T^{i}{}_{j})_{(a)} = (\mathcal{L}G^{i}{}_{al})\, T^{l}{}_{j} - (\mathcal{L}G^{l}{}_{aj})\, T^{i}{}_{l} - (\mathcal{L}G^{l}{}_{am})\, \dot{x}^m\, \dot{\partial}_l T^{i}{}_{j},$$

to $H^i{}_{jkl}$ and using (2.3), we have

$$\mathcal{L}(H^{i}{}_{jkl(a)}) - (\mathcal{L}H^{i}{}_{jkl})_{(a)} = - \dot{\partial}_a \dot{\partial}_p B^{iq}\, \sigma_q\, H^{p}{}_{jkl}$$

$$+ \dot{\partial}_a \dot{\partial}_j B^{pq}\, \sigma_q\, H^{i}{}_{pkl} + \dot{\partial}_a \dot{\partial}_k B^{pq}\, \sigma_q\, H^{i}{}_{jpl} + \dot{\partial}_a \dot{\partial}_l B^{pq}\, \sigma_q\, H^{i}{}_{jkp}$$

$$+ \dot{\partial}_a \dot{\partial}_m B^{pq}\, \sigma_q\, \dot{x}^m\, \dot{\partial}_p H^{i}{}_{jkl}. \qquad ...(3.6)$$

Transvecting (3.6) by $\dot{x}^a$ and noting that B^{pq} are homogeneous of second degree in their directional arguments, we have

$$\mathcal{L}(H^i_{jkl(a)})\,\dot{x}^a - (\mathcal{L}H^i_{jkl})_{(a)}\,\dot{x}^a = \sigma_q\,(-\,\dot{\partial}_p\,B^{iq}\,H^p_{jkl}$$

$$+\,\dot{\partial}_j\,B^{pq}\,H^i_{pkl} + \dot{\partial}_k\,B^{pq}\,H^i_{jpl} + \dot{\partial}_l\,B^{pq}\,H^i_{jkp} + F^2 g^{pq}\,\dot{\partial}_p\,H^i_{jkl}). \quad ...(3.7)$$

Since the transformation is such that it leaves H^i_{jkl} and $H^i_{jkl(a)}$ invariant, therefore

$$\mathcal{L}\,H^i_{jkl} = 0 \quad ...(3.8)$$

and

$$\mathcal{L}(H^i_{jkl(a)}) = 0. \quad ...(3.9)$$

Substituting (3.8) and (3.9) into (3.7), we have (3.5).

Theorem 3.4—If an H-symmetric Finsler space admits an infinitesimal conformal transformation under which the Berwald's curvature tensor H^i_{jkl} is invaraint, then the relation (3.5) holds.

PROOF : If the covariant derivative of the curvature tensor H^i_{jkl} vanishes, that is, $H^i_{jkl(a)} = 0$, then the condition (3.9) is always satisfied. Since this is the case for an H-symmetric Finsler space, therefore, the theorem follows from (3.7) on using (3.8) and (3.9).

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