ON EXISTENCE OF THE AFFINELY CONNECTED FINSLER SPACE WITH RECURRENT TENSOR FIELDS

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The present paper deals with the recurrent properties of the Berwald's curvature tensor field $H^i_{jkh}(x, \dot{x})$ by using Cartan's first and second covariant derivatives in an n-dimensional Finsler space F_n (Rund 1959). The recurrency in an affinely connected and projectively flat spaces have been studied.

1. INTRODUCTION

Let us consider an n-dimensional affinely connected Finsler space with a symmetric connection coefficient $\Gamma^{*i}_{jk} (= G^i_{jk})$, where $G^i_{jk} \overset{\text{def}}{=} \dot{\partial}_j \dot{\partial}_k G^i$, is the Berwald's connection parameter which is homogeneous of degree zero with respect to $\dot{x}^i$. Thus, the metric of this specified space will be such that $C_{ijk|h} = 0$. In view of which the directional derivative of Γ^{*i}_{jk} vanishes and hence $\dot{\partial}_1 G^i_{jk} = 0$. The tensor $A^k_{ih}(x, \dot{x})$ is given by

$$A^k_{ih} \overset{\text{def}}{=} FC^k_{ih}, \qquad \text{...(1.1)}$$

where

$$(a)\ A_{ijh} = g_{jk} A^k_{ih}, \qquad (b)\ A_{ijh}\dot{x}^i = 0. \qquad \text{...(1.2)}$$

The unit vector $l^i(x, \dot{x})$ is given by

$$l^i = \frac{\dot{x}^i}{F}. \qquad \text{...(1.3)}$$

The Berwald's curvature tensor $H^i_{hjk}(x, \dot{x})$ is defined by

$$H^i_{hjk}(x, \dot{x}) = \tfrac{2}{3}\, \dot{\partial}_h \dot{\partial}_{[j} H^i_{k]} \qquad \text{...(1.4)}$$

where

$$(a)\ H^i_{hjk} = \dot{\partial}_h H^i_{jk}, \quad (b)\ H^i_{hjk}\dot{x}^h = H^i_{jk}, \quad (c)\ H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0. \qquad \text{...(1.5)}$$

We have the following contractions:

$$(a)\ H^h{}_{ijh} = H_{ij} = \dot{\partial}_i H_j, \qquad (b)\ H^h{}_{ih} = H_i,$$

$$(c)\ H_{jh} - H_{hj} = H^s{}_{shj}, \qquad (d)\ H^i{}_{hj}\dot{x}^h = H^i{}_j,$$

$$(e)\ H^i{}_j \dot{x}^j = 0, \qquad (f)\ \dot{\partial}_l H^i{}_j \dot{x}^j = -H^i{}_l,$$

$$(g)\ H^i{}_i = (n-1) H. \qquad \ldots(1.6)$$

The Cartan's first and second covariant derivative of $X^i(x, \dot{x})$, are given by

$$(a)\ X^i{}_{|h} = \partial_h X^i - \dot{\partial}_k X^i \Gamma^{*k}_{jh} \dot{x}^j + X^k \Gamma^{*i}_{kh}$$

$$(b)\ X^i|_h = F \dot{\partial}_h X^i + A^i{}_{kh} X^k. \qquad \ldots(1.7)$$

We have

$$(a)\ l^i{}_{|h} = 0, \qquad (b)\ F_{|h} = 0 \qquad \ldots(1.8)$$

and

$$\dot{x}^i|_k = F\delta^i{}_k. \qquad \ldots(1.9)$$

The commutation formulae for vector $X^i(x, \dot{x})$ with respect to covariant derivative (1.7*a*) and (1.7*b*) are given by

$$(a)\ (\dot{\partial}_h X^i)_{|k} - \dot{\partial}_h X^i{}_{|k} = \dot{\partial}_l X^i A^l{}_{hk|r} l^r - \dot{\partial}_h \Gamma^{*i}_{rk} X^r,$$

$$(b)\ (\dot{\partial}_h X^i)|_k - \dot{\partial}_h X^i|_k = -\{F_{\dot{x}^h} \dot{\partial}_k X^i + \dot{\partial}_h A^i{}_{mk} X^m + A^m{}_{hk} \dot{\partial}_m X^i\}. \qquad \ldots(1.10)$$

The projective deviation tensor $W^i{}_j(x, \dot{x})$ (Rund 1959), is given by

$$W^i{}_j(x, \dot{x}) = H^i{}_j - H\delta^i{}_j - \frac{\dot{x}^i}{n+1} (\dot{\partial}_a H^a{}_j - \dot{\partial}_j H). \qquad \ldots(1.11)$$

2. Berwald's Recurrent Tensor Fields

Definition 2.1—An n-dimensional Finsler space F_n is said to be H^*-recurrent (Mishra and Pande 1968, Misra 1973) if the Berwald curvature tensor $H^i{}_{hjk}(x, \dot{x})$ satisfies the following relation:

$$H^i{}_{jkh|l} = v_l H^i{}_{jkh} \qquad \ldots(2.1)$$

where $v_l = v_l(x, \dot{x})$ is called the non-zero recurrence vector. Transvecting (2.1) by $\dot{x}^j$ and $\dot{x}^k$ successively, we get

$$H^i{}_{kh|l} = v_l H^i{}_{kh} \qquad \ldots(2.2)$$

and

$$H^i_{h|l} = v_l H^i_h. \qquad ...(2.3)$$

So that, the tensors H^i_{kh} and H^i_h are recurrent in H^*-recurrent F_n with the same recurrence vector v_l and are known as H^{*i}_{kh}-recurrent F_n and $H_h{}^{*i}$-recurrent F_n respectively. Contracting (2.1), (2.2) and (2.3) with respect to indices i, h and using relations (1.6a), (1.6b) and (1.6g) respectively, we get

$$H_{jk|l} = v_l H_{jk}, \qquad ...(2.4)$$

$$H_{k|l} = v_l H_k \qquad ...(2.5)$$

and

$$H_{|l} = v_l H. \qquad ...(2.6)$$

Hence H_{jk}, H_k and H are also recurrent in H^*-recurrent F_n. But the converse is not true. Differentiating (2.2) partially with respect to $\dot{x}^j$ and using (1.10a), we obtain

$$H^i_{jkh|l} - v_l H^i_{jkh} = H^i_{mkh} A^m_{jl|r} l^r - \dot{\partial}_j \Gamma^{*i}_{rl} H^r_{kh} + \dot{\partial}_j \Gamma^{*r}_{kl} H^i_{rh}$$

$$+ \dot{\partial}_j \Gamma^{*r}_{hl} H^i_{kr} + \dot{\partial}_j v_l H^i_{kh}. \qquad ...(2.7)$$

The equation (2.7) shows that the recurrence vector v_l will be homogeneous of degree zero with respect to $\dot{x}^i$ for $H^{*i}_{jk}(x, \dot{x})$-recurrent F_n. Hence, we have

Theorem 2.1—In an affinely connected F_n if the directional derivative of recurrence vector vanishes then H^{*i}_{jk}-recurrent F_n will be H^*-recurrent.

Commutating equation (2.2) with respect to indices k, h and l and adding all the three equations thus obtained, we get

Theorem 2.2—The Bianchi identity for H^i_{kh} reduces to the following form:

$$v_{[l} H^r{}_{kh]} = \partial_{[l} H^r{}_{kh]} + H^m_{[kh} \Gamma^{*r}_{l]m} - H^r{}_{m[kh} \Gamma^{*m}_{l]s} \dot{x}^s. \qquad ...(2.8)$$

Differentiating (1.11) covariantly with respect to x^l and using equations (1.11a), (2.3) and (2.6), we get

$$W^i_{j|l} - v_l W^i_j = -\frac{\dot{x}^i}{n+1} \{\dot{\partial}_a v_l H^a_j - \dot{\partial}_j v_l H + \dot{\partial}_m H^a_j A^m_{al|r} l^r$$

$$- \dot{\partial}_a \Gamma^{*a}_{rl} H^r_j - \dot{\partial}_a \Gamma^{*r}_{jl} H^a_r - \dot{\partial}_m H A^m_{jl|r} l^r\}. \qquad ...(2.9)$$

Contracting (2.9) with respect to indices i and j and using relations (1.6e), (1.6f) (1.1) and (1.3), we get

$$\dot{\partial}_j v_l \dot{x}^j H = H^a_m (C^m_{al|r} - \dot{\partial}_a \Gamma^{*m}_{lr}) \dot{x}^r. \qquad ...(2.10)$$

Since the right hand member of (2.10) vanishes in view of the condition $C^m_{al|r} \dot{x}^r = \dot{\partial}_a \Gamma^{*m}_{rl} \dot{x}^r$, then from here we see that the recurrence vector v_l is homogeneous of degree zero with respect to directional argument in H^*-recurrent F_n also. Then, summarising the above all results, we have the following theorems:

Theorem 2.3—In H^*-recurrent F_n the projective deviation tensor $W^i_j(x, \dot{x})$ is recurrent for the same recurrence vector v_l, if it satisfies the following relation:

$$\frac{\dot{x}^i}{n+1}\{\dot{\partial}_a v_l H^a_j - \dot{\partial}_j v_l H + \dot{\partial}_m H^a_j A^m_{a l|r} l^r$$

$$- \dot{\partial}_a \Gamma^{*a}_{rl} H^r_j - \dot{\partial}_a \Gamma^{*r}_{jl} H^a_r - \dot{\partial}_m H A^m_{jl|r} l^r\} = 0. \quad ...(2.11)$$

Theorem 2.4—The projective deviation tensor is recurrent in an affinely connected H^*-recurrent F_n, if it admits the following relation:

$$\frac{\dot{x}^i}{n+1}\{\dot{\partial}_a v_l H^a_j - \dot{\partial}_j v_l H\} = 0 \quad ...(2.12)$$

Theorem 2.5—In an affinely connected H^*-recurrent F_n the projective deviation tensor $W^i_j(x, \dot{x})$ is recurrent if the directional derivative of recurrence vector vanishes.

Theorem 2.6—In H^*-recurent F_n the recurrence vector is homogeneous of degree zero with respect to the directional argument.

Definition 2.2—An n-dimensional Finsler space F_n is said to be $\bar{H}$-recurrent (Pande and Singh 1974) if the Cartan's second covariant derivative of Berwald's curvature $H^i_{hjk}(x, \dot{x})$ satisfies the relation

$$H^i_{jkh}|_l = \lambda_l H^i_{jkh} \quad ...(2.13)$$

where $\lambda_l = \lambda_l(x, \dot{x})$ is a non-zero recurrence vector field. Contracting (2.13) with respect to indices i and h and using (1.6a), we get

$$H_{jk}|_l = \lambda_l H_{jk}. \quad ...(2.14)$$

Hence H_{jk} is recurrent in $\bar{H}$-recurrent F_n.

Differentiating (1.5b) covariantly with respect to x^l and using (1.9), we get

$$H^i_{kh}|_l = H^i_{jkh}|_l \dot{x}^j + FH^i_{lkh} \quad ...(2.15)$$

With the help of (2.13), equation (2.15) becomes

$$H^i_{kh}|_l - \lambda_l H^i_{kh} = FH^i_{lkh} \quad ...(2.16)$$

Further, differentiating (1.6d) covariantly with respect to x^l and using (1.9), we obtain

$$H^i_h|_l = H^i_{kh}|_l \dot{x}^k + FH^i_{lh}. \quad ...(2.17)$$

Using (2.16) in (2.17), we get

$$H^i_h|_l - \lambda_l H^i_h = F\eta^i_{lh} \quad ...(2.18)$$

where

$$\eta^i_{lh} \overset{\text{def}}{=\!=} H^i_{lh} + H^i_{lkh} \dot{x}^k. \quad ...(2.19)$$

Contracting (2.16) and (2.18) with respect to the indices i and h, we get

$$H_k \,|_l - \lambda_l H_k = F H_{lk} \qquad \text{...(2.20)}$$

and

$$H \,|_l - \lambda_l H = \frac{F}{n-1}\, \theta_l \qquad \text{...(2.21)}$$

where

$$\theta_l \overset{\text{def}}{=} \dot{\eta}^i{}_{li}. \qquad \text{...(2.22)}$$

Again, differentiating (1.6e) covariantly with respect to x^l and using the relations (1.9), (2.18), (1.6e), we get

$$F H^i{}_{lmk} \dot{x}^m \dot{x}^k = 0. \qquad \text{...(2.23)}$$

Thus, with the help of equations (2.16), (2.18), (2.20) and (2.21), we state the following theorem:

Theorem 2.7—The tensors $H^i{}_{jk}$ and $H^i{}_k$ are not recurrent for the same recurrence vector λ_l in $\bar{H}$-recurrent F_n.

Differentiating $H^r{}_{kh}$ covariantly with respect to x^l and adding all the three equations thus obtained by cyclic rotation of the indices l, k, h, we get

$$H^r{}_{[kh|l]} = F\dot{\partial}_{[l} H^r{}_{kh]} + A^r{}_{m\,[l} H^m_{kh]} - H^r{}_{m\,[h} A^m_{kl]} - A^m_{[hl} H^r{}_{]\,m}. \qquad \text{...(2.24)}$$

Multiplying (2.22) by $\dot{x}^h$ and using the homogeneity and skew-symmetric properties of $H^i{}_{kh}$ and the relations, (1.2b), (1.5c), we get

Theorem 2.8—In $\bar{H}$-recurrent F_n the recurrence vector λ_l satisfies the relation

$$2\lambda_{[k} H^r{}_{l]} + \lambda_h H^r{}_{lk} \dot{x}^h = 2A^r{}_{m\,[k} H^m_{l]}. \qquad \text{...(2.25)}$$

With the help of equations (2.18), (2.21) and the commutation formula (1.10b), we get

$$(\dot{\partial}_a H_j{}^a)\,|_l = \dot{\partial}_a \lambda_l H_j{}^a + \lambda_l \dot{\partial}_a H_j{}^a + \dot{\partial}_a (F\eta^a_{lj})$$
$$-\{F_{\dot{a}}\, a\dot{\partial}_l H_j{}^a + \dot{\partial}_a A^a_{ml} H_j{}^m - \dot{\partial}_a A^m_{jl} H^a_m + A^m_{al} \dot{\partial}_m H_j{}^a\} \qquad \text{...(2.26)}$$

and

$$(\dot{\partial}_j H)\,|_l = \dot{\partial}_j \lambda_l H + \lambda_l \dot{\partial}_j H + \frac{1}{n-1}\, \dot{\partial}_j (F\theta_l) - (F_{\dot{x}^j} \dot{\partial}_l H + A^m{}_{lj} \dot{\partial}_m H). \qquad \text{...(2.27)}$$

Differentiating (1.12) covariantly with respect to x^l and using the relations (2.18), (2.21), (2.26), (2.27) and (1.9), we obtain

$$W_j^i|_l - \lambda_l W_j^i = F\eta^i_{lj} - \frac{\delta^i_j}{n-1} F\theta_l - \frac{\dot{x}^i}{n+1} \Big\{\dot{\partial}_a \lambda_l H_j^a + \dot{\partial}_a (F\eta^a_{lj})$$

$$- (F\dot{x}^a \dot{\partial}_l H_j^a + \dot{\partial}_a A^a_{ml} H_j^m - \dot{\partial}_a A^m_{jl} H^a_m + A^m_{al} \dot{\partial}_m H_j^a)$$

$$- \dot{\partial}_j \lambda_l H - \frac{1}{n-1} \dot{\partial}_j (F\theta_l) + F\dot{x}^j \dot{\partial}_l H + A^m_{jl} \dot{\partial}_m H \Big\}$$

$$- F\delta_l^i (\dot{\partial}_a H^a_j - \dot{\partial}_j H)/(n+1). \qquad \ldots(2.28)$$

Accordingly, we have the following :

Theorem 2.9—In $\overline{H}$-recurrent F_n the projective deviation tensor $W_j^i(x, \dot{x})$ will be recurrent if the right hand member of (2.28) vanishes. Contracting (2.28) with respect to indices i, j and using equations (1.6*e*), (1.2*b*), (1.6*f*); we get

$$F(\theta_l + \dot{\partial}_a H_l^a + \dot{\partial}_a \eta^a_{li} \dot{x}^i) + F\dot{x}^i (H^a_l + \eta^a_{li} \dot{x}^i)$$

$$- \dot{\partial}_i \lambda_l \dot{x}^i H = 0. \qquad \ldots(2.29)$$

With the help of equations (2.19), (2.22) and (2.23) the equation (2.29) reduces to the form

$$\dot{\partial}_i \lambda_l \dot{x}^i H = F(2\theta_l + \dot{\partial}_a H_l^a - 3H_{li} \dot{x}^i). \qquad \ldots(2.30)$$

Thus, we have the following :

Theorem 2.10—In $\overline{H}$-recurrent F_n the recurrence vector λ_l is homogeneous of degree zero with respect to $\dot{x}^i$, when

$$F(2\theta_l + \dot{\partial}_a H_l^a - 3H_{li} \dot{x}^i) = 0. \qquad \ldots(2.31)$$

If it be possible that the tensors H^i_{jk}, H_k^i are recurrent in $\overline{H}$-recurrent F_n, then the terms $F\eta^i_{lh}$ and $F\theta_l$ vanish and the spaces are known as $\overline{H}^i_{jk}$-recurrent and $\overline{H}^i_k$-recurrent F_n respectively. Hence in view of this proposition, we have the following theorems:

Theorem 2.11—In $\overline{H}^i_{jk}$-recurrent F_n the Berwald's curvature tensor H^i_{jkl} is recurrent if the recurrence vector λ_l is independent of the directional argument and satisfies the following relation:

$$F\dot{x}^j \dot{\partial}_l H^i_{hk} + \dot{\partial}_j A^i_{ml} H^m_{hk} - \dot{\partial}_j A^m_{hl} H^i_{mk} - \dot{\partial}_j A^m_{kl} H^i_{hm}$$

$$+ A^m_{jl} \dot{\partial}_m H^i_{kh} = 0. \qquad \ldots(2.32)$$

Theorem 2.12—In $\overline{H}$-recurrent F_n the projective deviation tensor W_j^i is recurrent if the following relation is true:

$$\dot{x}^i \{\dot{\partial}_a \lambda_l H^a_j - (F\dot{x}^a \dot{\partial}_l H^a_j + \dot{\partial}_a A^a_{ml} H_j^m$$

$$- \dot{\partial}_a A^m_{jl} H^a_m + A^m_{al} \dot{\partial}_m H_j^a) - \dot{\partial}_j \lambda_l H$$

$$+ F\dot{x}^j \dot{\partial}_l H + A^m_{jl} \dot{\partial}_m H\} = 0 \qquad \text{for } i \neq l. \qquad \ldots(2.33)$$

REFERENCES

Misra, R. B. (1973). On a recurrent Finsler space. *Rev. Roum. Math. applic.*, **18** (5), 701–12.

Mishra, R. S., and Pande, H. D. (1968). Recurrent Finsler spaces. *J. Indian math. Soc.*, **32**, 17–22.

Pande, H. D., and Singh, B. (1974). On projective recurrent Finsler spaces of the first order. *Acad. Naz. Die*, **40**, 24–25.

Rund, H. (1959). The Differential Geometry of Finsler Spaces. Springer–Verlag, Berlin.