UNSTEADY MOTION OF VISCOELASTIC FLUID DUE TO ROTATION OF A SPHERE

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The unsteady motion of idealized viscoelastic fluid with relaxation time and retardation time parameters (Oldroyd's model 1950) due to slow rotation of a sphere about a diameter has been investigated by the application of Laplace transform. The case when the angular velocity of the sphere is uniform has been studied in detail. The solution for viscous fluid has been obtained as a limiting case.

1. INTRODUCTION

The problem of unsteady motion of a viscous incompressible fluid due to uniform rotation of a sphere about a diameter has been discussed by Ghildyal (1961). The analysis of the motion of an elastico-viscous liquid due to a sphere rotating about a diameter has been reported by Thomas and Walters (1964) and of the unsteady motion of a viscous fluid due to non-uniform rotation of a rough sphere by Gupta and Kulshreshtha (1970).

Recently, Tandon (1967) has studied the unsteady motion of an idealized viscoelastic fluid with relaxation time parameter λ_1 (Oldroyd's model 1950) due to the rotation of a sphere about a diameter. When the sphere rotates with uniform angular velocity, Tandon has observed that the disturbance is not instantaneous. However, in his study, Tandon has ignored the effect of the retardation time parameter λ_2. The object of the present paper is to consider the more general situation when the effects of both relaxation time and retardation time parameters (λ_1, λ_2) are taken into consideration. In the present case it has been observed that the disturbance is instantaneous and the velocity of propagation of the disturbance is finite, as must be expected from physical considerations.

2. FORMULATION OF THE PROBLEM

For slow motion, the equations of state relating the stress tensor s_{ik} and the rate of strain tensor $e_{ik} = \frac{1}{2}(v_{k,i} + v_{i,k})$ of the idealized fluids are of the form (Frater, 1967)

$$s_{ik} = p_{ik} - pg_{ik} \qquad ...(2.1)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p_{ik} = 2\eta_0 \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) e_{ik}. \qquad ...(2.2)$$

Here p_{ik} is the part of the stress-tensor related to the change of shape of the material element; g_{ik} the metric tensor; p an isotropic pressure; and v_i the velocity vector; λ_1 and λ_2 are constants with dimensions of time; η_0 the coefficient of viscosity $(\lambda_1 \geqslant \lambda_2 > 0, \eta_0 > 0)$.

The equations of motion, when linearized, and the equation of continuity are given by

$$\rho \frac{\partial v^i}{\partial t} = - g^{ik} p_{,k} + p^{ik}{}_{,k} \qquad ...(2.3)$$

$$e^j{}_j = 0 \qquad ...(2.4)$$

Consider unsteady motion of a viscoelastic fluid surrounding a solid sphere of radius a which is made to rotate about a diameter with an angular velocity which is an arbitrary function of time. The fluid is assumed to be at rest initially. We take spherical polar coordinates (r, θ, ϕ) with the centre of the sphere as the origin and the axis of rotation as the z-axis. From consideration of symmetry, the velocity field will be independent of ϕ. Thus, for the flow under consideration, the physical components of velocity are given by

$$v_r = 0, \qquad v_\theta = 0, \qquad v_\phi = r \sin\theta\, \omega(r, t). \qquad ...(2.5)$$

Using (2.5) and transforming (2.2) to polar coordinates, we have

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p_{rr} = 0 \qquad ...(2.6)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p_{\theta\theta} = 0 \qquad ...(2.7)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p_{\phi\phi} = 0 \qquad ...(2.8)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p_{r\theta} = 0 \qquad ...(2.9)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p_{r\phi} = \eta_0 \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left(r \sin\theta \frac{\partial \omega}{\partial r}\right) \qquad ...(2.10)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p_{\theta\phi} = 0 \qquad ...(2.11)$$

From equations (2.6) through (2.9) and (2.11), we have

$$p_{rr} = p_{\theta\theta} = p_{\phi\phi} = p_{r\theta} = p_{\theta\phi} = 0. \qquad ...(2.12)$$

Hence from (2.3), the three equations of motion reduce to

$$0 = - \frac{1}{\rho} \frac{\partial p}{\partial r} \qquad ...(2.13)$$

$$0 = - \frac{1}{\rho} \frac{\partial p}{r \partial \theta} \qquad ...(2.14)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial \omega}{\partial t} = \nu \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left\{\frac{\partial^2 \omega}{\partial r^2} + \frac{4}{r} \frac{\partial \omega}{\partial r}\right\} \qquad ...(2.15)$$

where ν $(= \eta_0/\rho)$ is the kinematic viscosity.

From (2.13) and (2.14), we find that the pressure remains constant throughout the liquid.

Now, the problem is to solve eqn. (2.15) under the following initial and boundary conditions:

$$\left.\begin{aligned} &\text{(i) } \omega(r, 0) = 0 = (\partial \omega/\partial t)_{r,0} \\ &\text{(ii) } \omega(a, t) = f(t) \\ &\text{(iii) } \lim_{r \to \infty} \omega(r, t) = 0. \end{aligned}\right\} \qquad ...(2.16)$$

3. Solution of the Problem

Putting $\omega = w/r^{3/2}$ in (2.15) and (2.16), we obtain

$$\lambda_1 \frac{\partial^2 w}{\partial t^2} + \frac{\partial w}{\partial t} = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - \frac{9}{4} \cdot \frac{w}{r^2}\right\} \qquad ...(2.17)$$

and

$$\left.\begin{aligned} &\text{(i) } w(r, 0) = 0 = (\partial w/\partial t)_{r,0} \\ &\text{(ii) } w(a, t) = a^{3/2} f(t) \\ &\text{(iii) } \lim_{r \to \infty} w(r, t) = 0. \end{aligned}\right\} \qquad ...(2.18)$$

Now we define the Laplace transform of $w(r, t)$ as

$$\bar{w}(r, s) = \int_0^\infty w(r, t)\, e^{-st}\, dt \qquad ...(2.19)$$

Multiplying both sides of (2.17) by e^{-st} and integrating between the limits $t = 0$ to $t = \infty$, and using conditions (i) of (2.18), we obtain

$$\frac{d^2 \bar{w}}{dr^2} + \frac{1}{r} \frac{d\bar{w}}{dr} - \left\{\frac{9}{4r^2} + \frac{s(\lambda_1 s + 1)}{\nu(\lambda_2 s + 1)}\right\} \bar{w} = 0. \qquad ...(2.20)$$

Also, the conditions (2.18) transform to

(i) $\bar{w}(r, 0) = 0 = (\partial \bar{w}/\partial s)_{r,0}$

(ii) $\bar{w}(a, s) = a^{3/2} \bar{f}(s)$

(iii) $\lim_{r \to \infty} \bar{w}(r, s) = 0$ ...(2.21)

The solution of (2.20) is

$$\bar{w}(r, s) = A I_{3/2}(rq) + B K_{3/2}(rq) \qquad ...(22\ 2)$$

where $I_{3/2}$ and $k_{3/2}$ are modified Bessel functions of order 3/2 and

$$q = \sqrt{\frac{s(\lambda_1 s + 1)}{\nu(\lambda_2 s + 1)}}. \qquad \text{...(2.23)}$$

The boundary conditions give

$$A = 0 \quad \text{and} \quad B = a^{3/2}\bar{f}(s)/K_{3/2}(aq).$$

Therefore,

$$\bar{w}(r, s) = a^{3/2}\frac{K_{3/2}(rq)}{K_{3/2}(aq)}\bar{f}(s).$$

For the order $(n + \frac{1}{2})$, where n is an integer, the Bessel functions and modified Bessel functions both reduce to finite form. Thus on using the result (McLachlan 1955)

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \sum_{r=0}^{n} \frac{(n+r)!}{r!\,(n-r)!\,(2z)^r}$$

we get

$$\bar{w}(r, s) = \frac{a^3}{r^{3/2}} \cdot \frac{1 + rq}{1 + aq} \cdot e^{-(r-a)q}\,\bar{f}(s). \qquad \text{...(2.24)}$$

From Laplace inversion formula, this gives

$$\omega(r, t) = \frac{a^3}{r^3} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st}\,\bar{f}(s) \cdot \frac{1 + rq}{1 + aq}\, e^{-(r-a)q}\, ds \qquad \text{...(2.25)}$$

where c is greater than the real part of all the singularities of the integrand.

3. A Particular Case

Consider the unsteady flow of viscoelastic liquid initially at rest due to the uniform rotation, Ω, of the sphere about the z-axis. In the present case, we have

$$f(t) = \Omega$$

so that

$$\bar{f}(s) = \frac{\Omega}{s}.$$

Substituting this in (2.25), we see that the required disturbance is given by

$$\omega(r, t) = \frac{a^3}{r^3} \cdot \frac{\Omega}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s} \cdot \frac{1 + r\sqrt{\dfrac{s(\lambda_1 s + 1)}{\nu(\lambda_2 s + 1)}}}{1 + a\sqrt{\dfrac{s(\lambda_1 s + 1)}{\nu(\lambda_2 s + 1)}}}$$

$$\times \exp\left[-(r-a)\left(\frac{s(\lambda_1 s + 1)}{\nu(\lambda_2 s + 1)}\right)^{\frac{1}{2}}\right] ds. \qquad \text{...(3.1)}$$

The singularities of the integrand are the branch points

$$s = 0, \quad s = -\frac{1}{\lambda_1}, \quad s = -\frac{1}{\lambda_2}.$$

Thus by considering Bromwich contour with a cut along the negative real axis and indentations at $s = 0, s = -1/\lambda_1, s = -1/\lambda_2$ and evaluating the integral on the right of (3.1), we obtain

$$\omega(r,t) = \frac{a^3\Omega}{r^3}\Bigg[1 - \frac{1}{\pi}\int_{1/\lambda_2}^{\infty} \frac{e^{-\sigma t}}{\sigma}\cdot\frac{1 + ar\left\{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}\right\}}{1 + a^2\left\{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}\right\}}$$

$$\times \sin(r-a)\sqrt{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}}\,d\sigma$$

$$+ \frac{1}{\pi}\int_{1/\lambda_2}^{\infty} \frac{e^{-\sigma t}}{\sigma}\cdot\frac{(r-a)\sqrt{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}}}{1 + a^2\left\{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}\right\}}$$

$$\times \cos(r-a)\sqrt{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}}\,d\sigma$$

$$- \frac{1}{\pi}\int_{0}^{1/\lambda_1} \frac{e^{-\sigma t}}{\sigma}\cdot\frac{1 + ar\left\{\frac{\sigma(1 - \lambda_1\sigma)}{\nu(1 - \lambda_2\sigma)}\right\}}{1 + a^2\left\{\frac{\sigma(1 - \lambda_1\sigma)}{\nu(1 - \lambda_2\sigma)}\right\}}$$

$$\times \sin(r-a)\sqrt{\frac{\sigma(1 - \lambda_1\sigma)}{\nu(1 - \lambda_2\sigma)}}\,d\sigma$$

$$+ \frac{1}{\pi}\int_{0}^{1/\lambda_1} \frac{e^{-\sigma t}}{\sigma}\cdot\frac{(r-a)\sqrt{\frac{\sigma(1 - \lambda_1\sigma)}{\nu(1 - \lambda_2\sigma)}}}{1 + a^2\left\{\frac{\sigma(1 - \lambda_1\sigma)}{\nu(1 - \lambda_2\sigma)}\right\}}$$

$$\times \cos(r-a)\sqrt{\frac{\sigma(1 - \lambda_1\sigma)}{\nu(1 - \lambda_2\sigma)}}\,d\sigma\Bigg] \qquad \ldots(3.2)$$

4. Discussion

The value of $\omega(r,t)$ is given by the expression (3.2) and it satisfies the governing equation of motion (2.15) as well as the boundary conditions (2.16). This is quite a new result for the disturbance of an idealized viscoelastic liquid with three parameters due to the uniform rotation of a sphere about a diameter. From (3.2), it is apparent that like viscous fluid the disturbance is instantaneous and the velocity of propagation of the disturbance is finite.

5. Limiting Cases

When the viscoelastic parameters $\lambda_1 \to 0$, $\lambda_2 \to 0$, the liquid will behave as ordinary viscous liquid.

Now in the limit when $\lambda_1 \to 0$, $\lambda_2 \to 0$, it is easy to see that

$$\int_{1/\lambda_2}^{\infty} \frac{e^{-\sigma t}}{\sigma} \cdot \frac{1 + ar\left\{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}\right\}}{1 + a^2\left\{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}\right\}} \sin(r - a)\sqrt{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}}\, d\sigma \to 0 \qquad \ldots(5.1)$$

and

$$\int_{1/\lambda_2}^{\infty} \frac{e^{-\sigma t}}{\sigma} \cdot \frac{(r - a)\sqrt{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}}}{1 + a^2\left\{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}\right\}} \cos(r - a)\sqrt{\frac{\sigma(\lambda_1\sigma - 1)}{\nu(\lambda_2\sigma - 1)}}\, d\sigma \to 0. \ldots(5.2)$$

Hence from (3.2), we see that the corresponding viscous solution is given by

$$\omega(r, t) = \frac{a^3\Omega}{r^3}\Bigg[1 - \frac{1}{\pi}\int_0^{\infty} \frac{e^{-\sigma t}}{\sigma} \cdot \frac{1 + \frac{ar}{\nu}\sigma}{1 + \frac{a^2}{\nu}\sigma} \sin(r - a)\sqrt{\frac{\sigma}{\nu}}\, d\sigma$$

$$+ \frac{r - a}{\pi}\int_0^{\infty} \frac{e^{-\sigma t}}{\sigma} \cdot \frac{\sqrt{\sigma/\nu}}{1 + a^2\sigma/\nu} \cos(r - a)\sqrt{\frac{\sigma}{\nu}}\, d\sigma\Bigg]. \qquad \ldots(5.3)$$

Making use of certain integrals (Sneddon 1951), this reduces to

$$\omega(r, t) = \frac{a^3\Omega}{r^3}\Bigg[\left(\frac{r}{a} - 1\right)\exp\left(\frac{r - a}{a} + \frac{\nu t}{a^2}\right)\left\{1 - \operatorname{erf}\left(\frac{r - a}{2\sqrt{\nu t}} + \frac{\sqrt{\nu t}}{a}\right)\right\}$$

$$+ 1 - \operatorname{erf}\left(\frac{r - a}{2\sqrt{\nu t}}\right)\Bigg]. \qquad \ldots(5.4)$$

This is in complete agreement with the result obtained by Ghildyal (1961). Again, if we let $t \to \infty$, (3.2) reduces to

$$\omega(r, t) = \frac{a^3\Omega}{r^3} \qquad \ldots(5.5)$$

showing that in the steady state the liquid behaves like viscous liquid and the result (5.5) agrees with the steady state solution given in Lamb (1932).

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REFERENCES

Frater, K. R. (1967). Drag on a sphere oscillating in an elastico-viscous fluid. *Z. angew Math. Phys.* **18**, 798–803.

Ghildyal, C. D. (1961). Unsteady motion of an infinite liquid due to the uniform rotation of a sphere. *Quart. appl. Math.*, **18** (4), 396-99.

Gupta, P. M., and Kulshreshtha, S. K. (1970). Unsteady motion of an infinite liquid due to non-uniform rotation of a rough sphere. *Bull. Calcutta math. Soc.*, **62**, 19–24.

Lamb, H. (1932). Hydrodynamics. Cambridge University Press.

McLachlan, N. W. (1955). Bessel Functions for Engineers, Oxford University Press, London.

Oldroyd, J. G. (1950). *Proc. R. Soc.*, **A 200**, 523.

Sneddon, I. N. (1951). Fourier Transforms. McGraw-Hill Book Co., Inc., New York.

Tandon, P. N. (1967). Non-steady slow rotation of a sphere in a viscoelastic fluid. Ph.D. thesis, Agra University, Agra.

Thomas, R. H., and Walters, K. (1964). The motion of an elastico-viscous liquid due to a sphere rotating about its diameter. *Q. Jl Mech. appl. Math.*, **17**, 39.