PARTIAL KRONECKER PRODUCT AND ITS APPLICATIONS IN MODULAR HADAMARD MATRICES*

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Three new matrix products—row and column Schur products, and partial Kronecker product—of two matrices of the same order are defined. Some properties of partial Kronecker product are established. As an application of partial Kronecker product, a method of construction of *MHM* (mod n)'s is described.

1. INTRODUCTION

Throughout, the matrices are considered over a field. By an $(a_1, \ldots, a_n)$-matrix we mean a matrix with entries equal to $a_1, \ldots,$ or a_n. Let I_n be an identity matrix of order n, and $J_{m,n}$ be the matrix of order m by n with all its entries unities. Whenever there is no confusion the suffixes m and n may be omitted.

An 'Hadamard matrix' (H-matrix) H is a $(-1, 1)$-matrix of order h such that any two distinct rows of H are orthogonal. That is, H satisfies

$$HH^T = hI_h = H^TH$$

where A^T stands for transpose of A.

These matrices are conjectured to exist for the orders $h = 1$ or 2, or $h \equiv 0$ (mod 4). Though a number of infinite series of H-matrices are known to date, the problem of constructing H-matrices for all orders $h \equiv 0$ (mod 4) remains open. The minimum order for which H-matrix is unknown is 188. For a survey article and other terminology the reader is referred to Wallis *et al.* (1972).

Marrero and Butson (1972, 1973) have generalized the concept of H-matrix to modular Hadamard matrix (mod n) (*MHM* (mod n)). A $(-1, 1)$-matrix H of order h is said to be '*MHM* (mod n)' if any two distinct rows of H are orthogonal (mod n). For matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same order, $A \equiv B$ (mod n) if and only if $a_{ij} \equiv b_{ij}$ (mod n). Then, *MHM* (mod n) H satisfies.

$$HH^T \equiv hI_h \pmod{n}.$$

Henceforth, such a matrix will be designated by an $H(n, h)$-matrix.

Some special classes of *MHM* (mod n)'s may be employed in the construction of H-matrices (for example, see Hebbare and Patwardhan 1976).

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In this paper, we have defined three new matrix products—row and column Schur products, and partial Kronecker product, of two matrices of the same order. Some properties of partial Kronecker product are established. As an application of partial Kronecker product, an infinite series of MHM (mod n)'s is constructed.

Applications of partial Kronecker product in the class of (0, 1)-matrices will be considered in the forthcoming papers.

2. Three New Matrix Products

In this section definitions and some properties of row and column Schur products, and partial Kronecker product of two matrices of the same order are discussed.

Row and column Schur products—Let A and B be matrices of order m by n. Let $A = [A_1 A_2 \ldots A_m]^T$ and $B = [B_1 B_2 \ldots B_m]^T$ where A_i^T $(i = 1, \ldots, m)$ is row i of A and B_j^T $(j = 1, \ldots, m)$ is row j of B. Then 'row Schur product' $R(A, B)$ of A and B is of order $\binom{m}{2}$ by n defined by

$$R(A, B) = [A_1 \circ B_2 \ldots A_1 \circ B_m \, A_2 \circ B_3 \, A_2 \circ B_4 \ldots A_2 \circ B_m \ldots A_{m-1} \circ B_m]^T$$

where rows A_i and B_j $(i < j)$ are selected in all possible ways and arranged in lexicographic order and $a \circ b$ is the Schur product, that is, component-wise multiplication of the vectors a and b.

Clearly, if $A = B$ then $R(A, B) = R(B, A)$. In this case $R(A, A)$ is called 'row Schur matrix' $R(A)$ of A. 'Column Schur product' $C(A, B)$ of A and B and 'Column Schur matrix' $C(A)$ of A each of order m by $\binom{n}{2}$ may be defined similarly.

Row and Column Schur matrices have been used to construct new block designs from a given block design (see Hebbare *et al.* 1976*a*).

Partial Kronecker product—Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices each of order m by n. Then 'partial Kronecker product' $A * B$ of A and B is a matrix of order $m(m-1)$ by $n(n-1)$ defined by

$$A * B = (a_{ij} B_{ij})$$

where B_{ij} stands for the 'cofactor matrix' of b_{ij}—the submatrix of B obtained by deleting row i and column j. Each B_{ij} is of order $m-1$ by $n-1$.

Then it can be verified that, there exist suitable permutation matrices P and Q of orders m^2 and n^2 respectively, such that

$$A \times B = P \left[\begin{array}{c|cc} A \circ B & C(A, B) & C(B, A) \\ \hline R(A, B) & & \\ R(B, A) & & A * B \end{array}\right] Q \qquad \ldots(2.1)$$

where $A \times B$ and $A \circ B$ denote the Kronecker product and Schur product of A and B respectively. If A and B are of order m each then $Q = P^T$.

Some of the properties satisfied by '$*$' operation are given below. They follow from corresponding properties satisfied by Kronecker products and (2.1):

(i) $A * B \sim B * A$, where $D \sim E$ denotes that D and E are permutationally equivalent,

(ii) if $A = B$ then $A * B = B * A$,

(iii) $(A + B) * C = A * C + B * C$ and $C * (A + B) = C * A + C * B$,

(iv) $(A * B)^T = A^T * B^T$,

(v) $tr\,(A \times B) = tr\,(A * B) + tr\,(A \circ B)$,

where $tr\,(A)$ denotes trace of A.

We observe that $A * B$ is the principal submatrix of $A \times B$, that is the entries of $A * B$ are symmetrically situated along the principal diagonal. Furthermore partial Kronecker product, row and column Schur products and the usual matrix multiplication are connected by the following identity:

$$(A * B)(D * E) = AD * BE - \begin{bmatrix} R(A, B) \\ R(B, A) \end{bmatrix} [C(D, E) \quad C(E, D)] \qquad \ldots(2.2)$$

where A, B, D and E are conformable matrices.

This follows easily since under Kronecker products, we have

$$(A \times B)(D \times E) = AD \times BE \qquad \ldots(2.3)$$

and from (2.1) and (2.3) we have

$$\begin{bmatrix} * & * \\ * & M + (A * B)(D * E) \end{bmatrix} = \left[\begin{array}{c:cc} AD \circ BE & C(AD, BE) & C(BE, AD) \\ \hdashline R(AD, BE) & & \\ \hdashline R(BE, AD) & \multicolumn{2}{c}{AD * BE} \end{array}\right]$$

where

$$M = \begin{bmatrix} R(A, B) \\ R(B, A) \end{bmatrix} [C(D, E) \quad C(E, D)]$$

and the terms in the positions '$*$' are unspecified.

In particular, when $B = I_n = D$ and A, E are of order n, from (2.2) we have

$$(A * I_n)(I_n * E) = A * E - \begin{bmatrix} R(A, I_n) \\ R(I_n, A) \end{bmatrix} [C(I_n, E) \quad C(E, I_n)]. \qquad \ldots(2.4)$$

Generally, an explicit formula for det $(A * E)$-determinant of $A * E$ is difficult to obtain. However, when A and E are such that the subtracted matrix on the right-hand side of (2.4) is zero, then

$$\det(A * E) = \prod_{i=1}^{n} \det(A_{ii}) \det(E_{ii}).$$

The requirement that $M = 0$ when $B = I_n = E$ is not uncommon, for example if

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } E = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

then a simple computation shows that

$$(A * I_n) \quad (I_n * E) = A * E.$$

In fact, $M = 0$ also holds if either of A and E is diagonal and $B = I_n = D$.

3. A Construction of *MHM*'s Using Partial Kronecker Product

Theorem 3.1—An $H(n, v(v-1))$-matrix can be constructed from an irreducible SBIBD with the parameters $(v, v-1, v-2)$ where

$$n = \text{g.c.d.}\{v^2 - v - 8, \quad v^2 - v - 12, \quad v^2 - v - 16, \\ v^2 - 9v + 16, \quad v^2 - 9v + 20, \quad v^2 - 9v + 24\}.$$

Proof: Let N be the incidence matrix of the given irreducible SBIBD. Then $N = J - I$ with suitable relabelling of symbols and blocks of the SBIBD if necessary. Consider, $A = N - I$ which, in fact, is an $H(v-4, v)$-matrix satisfying

$$A^T A = 4I_v + (v-4) J_{v,v} = AA^T.$$

From (2.2) we have

$$(A * A)^T (A * A) = A^T A * A^T A - (R(A) R^T(A)) X J_{2,2}.$$

Now, let

$$R(A) R^T(A) = (r_{ij}).$$

When $i \neq j$, let $i \sim \alpha \circ \beta$ and $j \sim \gamma \circ \delta$. That is, row i of $R(A)$ is obtained by the Schur product of rows α and β of A and similarly, column j of $R^T(A)$ is obtained by the Schur product of rows γ and δ of A. Note that either all four symbols α, β, γ and δ are distinct or exactly three of them are distinct.

From the definition of A, we claim that

$$r_{ij} = \begin{cases} v-4, & \text{if three of } \alpha, \beta, \gamma, \delta \text{ are distinct} \\ v-8, & \text{otherwise.} \end{cases}$$

In the former case, let $\alpha = \gamma$, that is $i \sim \alpha \circ \beta$ and $j \sim \alpha \circ \delta$. Then inner product of $\alpha \circ \beta$ and $\alpha \circ \delta$ is given by

$$(\alpha \circ \beta, \quad \alpha \circ \delta) = \sum_{i=1}^{v} \alpha_i^2 \beta_i \delta_i$$

$$= \sum_{i=1}^{v} \beta_i \delta_i$$

$$= (\beta, \delta) = v - 4,$$

where $\alpha = (\alpha_1 \ldots \alpha_v)$, $\beta = (\beta_1 \ldots \beta_v)$ and $\delta = (\delta_1 \ldots \delta_v)$, and (α, β) denotes the inner product of vectors α and β.

In the latter case, without loss of generality, the rows labelled α, β, γ and δ may be arranged as shown below:

$$\begin{array}{lcccc}
\alpha : 1 \ldots 1 & 1 & 1 & 1 & -1 \\
\beta : 1 \ldots 1 & 1 & 1 & -1 & 1 \\
\gamma : 1 \ldots 1 & 1 & -1 & 1 & 1 \\
\delta : \underbrace{1 \ldots 1}_{v-4} & -1 & 1 & 1 & 1
\end{array}$$

Hence $(\alpha \circ \beta, \gamma \circ \delta) = v - 8$. Thus $R(A)\ R^T(A)$ has either $v - 4$ or $v - 8$ in the off-diagonal positions. Also

$$A^T A * A^T A = 16\, I + 4\,(v - 4)\,(I * J + J * I) + (v - 4)^2 J * J.$$

That is, $A^T A * A^T A$ has $(v - 4)^2$ or $v^2 - 16$ in the off-diagonal positions.

Thus $A * A$ has the required inner products as in (3.1). This completes the proof of the theorem.

Corollary 3.2—If $v \equiv 0 \pmod 4$ then $A * A$ is an $H(4, v(v - 1))$-matrix.

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