UNSTEADY FLOW OF A DUSTY VISCOUS FLUID THROUGH A TUBE WITH SECTOR OF A CIRCLE AS CROSS-SECTION

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The unsteady laminar flow of an incompressible viscous fluid with uniform distribution of dust particles through a tube whose cross-section is a sector of a circle subtending an angle 2α at the centre, under the influence of exponential pressure gradient with respect to time, has been investigated. The influence of the dust particles on the fluid is discussed and it is found that the velocity of dust particles is greater than that of the fluid.

INTRODUCTION

Interest in problems of mechanics of systems with more than one phase has developed rapidly in recent years. Situations which occur frequently are concerned with the motion of a liquid or gas which contains a distribution of solid particles. Such situations occur for example, in the movement of dust-laden air, in problems of fluidization, in the use of dust in gas-cooling systems to enhance heat transfer processes and in the process by which rain drops are formed by coalescence of small droplets which might be considered as solid particles for the purpose of examining their movement prior to coalescence.

The mathematical description of such diverse systems must of course vary widely. In problems of fluidization, for example, the bulk concentration of the dust particles is large, whereas in the problems of dust flow this will be small. The factor may bring considerable simplification to the theory of dusty gas flows, since the effect of one particle on another is not so pronounced and a good approximation might be expected by assuming the motion of one solid particle not being influenced by the surrounding particles which will be, in general, many particle diameters away.

In order to formulate the problem in a reasonably simple manner and to bring out the essential features we shall make simplifying assumptions about the motion of the dust particles. It will be supposed that the dust particles are uniform in size and shape, and their velocity and number density can be described by fields $\vec{u}(\vec{x}, t)$ and $N(\vec{x}, t)$. We also assume that the bulk concentration (i.e. concentration by volume) of the dust is very small so that the net effect of the dust on the fluid is equivalent to an extra force $KN(\vec{v}-\vec{u})$ per unit volume, where $\vec{u}(\vec{x}, t)$ is the velocity of the fluid and K is a constant, where it is also supposed that the Reynolds number of the relative motion of dust and fluid is small compared with unity so that the force between dust and fluid is proportional to the relative velocity.

Much work has already been done on such models of dusty fluid flow. Michael and Miller (1966) have discussed the motion of dusty gas occupying the semi-infinite space above a rigid plane boundary. Sambasiva Rao (1969) has studied the flow of a dusty viscous liquid through circular cylinder by taking exponential pressure gradient with respect to time. With similar pressure gradient Reddy (1972) has investigated the case of rectangular channel. Grishwar Nath (1970) has studied the laminar flow of an unsteady incompressible viscous fluid with uniform distribution of dust particles through two rotating coaxial cylinders under the influence of an axial pressure gradient. In the present paper the authors, taking exponential pressure gradient with respect to time, have made an attempt to study the laminar flow of an unsteady viscous fluid with uniform distribution of dust particles, through a circular tube whose cross-section is a sector of a circle subtending an angle 2α at the centre. It is interesting to note that the velocity of the dust particles is greater than that of fluid.

Governing Equations and Solution

The equations of motion of an unsteady viscous fluid with uniform distribution of dust particles given by Saffman (1962) are the following :

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\ \nabla)\ \vec{u}\right) = -\ \overline{\text{grad}}\ \vec{p} + \eta\, \nabla^2 \vec{u} + KN(\vec{v} - \vec{u}) \qquad \ldots(1)$$

$$\text{div}\ \vec{u} = 0 \qquad \ldots(2)$$

$$m\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\ \nabla)\ \vec{v}\right) = K(\vec{u} - \vec{v}) \qquad \ldots(3)$$

$$\frac{\partial N}{\partial t} + \text{div}\ N\ \vec{v} = 0 \qquad \ldots(4)$$

where $\vec{u}$ and $\vec{v}$ are the velocities of fluid and dust respectively, N is the number density of dust particles, each of mass m, K is the Stokes-coefficient of resistance and p, ρ, η the pressure, density and viscosity of the fluid. The time relaxation parameter $\tau = m/K$ (i.e. time for the dust to adjust to changes in fluid velocity) is given by (3). As both m and ρ are uniform N can be taken as constant and equal to N_0.

We take for reference frame a cylindrical polar system of coordinates (r, θ, z), the z-axis being taken along the length of the tube through which the flow is to be considered. The cross-section of the tube being a sector bounded by two radii $\theta = \pm\, \alpha$ and the circle $r = a$. For the present problem the velocity distribution of fluid and dust particles are defined respectively as

$$\left.\begin{array}{lll} u_1 = 0, & u_2 = 0, & u_3 = u_3(r, \theta, t) \\ v_1 = 0, & v_2 = 0, & v_3 = v_3(r, \theta, t) \\ & N = N_0 \text{ (a constant)} & \end{array}\right\} \quad \ldots(5)$$

where (u_1, u_2, u_3) and (v_1, v_2, v_3) are velocity components of fluid and dust particles respectively. The equations of motion then reduce to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \ldots(6)$$

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \quad \ldots(7)$$

$$\frac{\partial u_3}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2}\right) + \frac{KN_0}{\rho}(v_3 - u_3) \quad \ldots(8)$$

$$m\frac{\partial v_3}{\partial t} = K(u_3 - v_3). \quad \ldots(9)$$

From (6) and (7), it follows that $(-\partial p/\rho\partial z)$ is a function of t only. Since we have assumed the pressure gradient to be exponential, we can write

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = Ce^{-\lambda^2 t} \quad \ldots(10)$$

where C and λ are real constants.

In view of (10), we can express

$$u_3(r, \theta, t) = w_1(r, \theta)\, e^{-\lambda^2 t} \quad \ldots(11)$$

$$v_3(r, \theta, t) = w_2(r, \theta)\, e^{-\lambda^2 t}. \quad \ldots(12)$$

Using (10) to (12) in (8) and (9), we obtain respectively

$$\frac{\partial w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2} + \frac{l}{\nu\tau}(w_2 - w_1) + \frac{\lambda^2}{\nu}w_1 + \frac{C}{\nu} = 0, \quad \ldots(13)$$

and

$$w_2 = w_1 / (1 - \lambda^2\tau) \quad \ldots(14)$$

where $l = mN_0/\rho$ and $\tau = m/K$.

Eliminating w_2 from (13) and (14) we have the following equation:

$$\frac{\partial w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial w_1}{\partial \theta^2} + \frac{\lambda^2}{\nu}\left(\frac{1 + l - \lambda^2\tau}{1 - \lambda^2\tau}\right) \times \left[w_1 + \frac{C}{\lambda^2}\right]\left(\frac{1 - \lambda^2\tau}{1 + l - \lambda^2\tau}\right) = 0 \quad \ldots(15)$$

which can be simplified as

$$\frac{\partial w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2} + \beta^2 (w_1 + \Omega) = 0 \qquad ...(16)$$

where

$$\Omega = \frac{C}{\lambda^2}\left(\frac{1 - \lambda^2\tau}{1 + l - \lambda^2\tau}\right)$$

$$\beta^2 = \frac{\lambda^2}{\nu}\left(\frac{1 + l - \lambda^2\tau}{1 - \lambda^2\tau}\right).$$

The no slip boundary conditions are

$$\left.\begin{aligned} w_1 (r, \theta) &= 0 \text{ when } r = a, \quad -\alpha \leqslant \theta \leqslant \alpha \\ w_1 (r, \theta) &= 0 \text{ when } \quad \theta = \pm \alpha, \quad 0 \leqslant r \leqslant a \end{aligned}\right\} \qquad ...(17)$$

Let us assume that the solution is of the form

$$w_1 = \sum_{n=1}^{\infty} \{A_{k_n}(r) \cos k_n\theta + B_{k_n}(r) \sin k_n\theta\} \qquad ...(18)$$

where k_n is some function of integer n.

With the help of (17) we can choose the transformation as

$$\bar{w}_1 = \int_0^{\alpha} w_1 \cos \frac{2m+1}{2\alpha} \pi\theta d\theta \qquad ...(19)$$

Applying it to eqn. (16), we get

$$\frac{d^2 \bar{w}_1}{dr^2} + \frac{1}{r}\frac{d\bar{w}_1}{dr} - \frac{\mu^2}{r^2}\bar{w}_1 + \beta^2\left(\bar{w}_1 + \frac{(-1)^m \Omega}{\mu}\right) = 0 \qquad ...(20)$$

where

$$\mu = (2m + 1)\ \pi/2\alpha.$$

Applying finite Hankel transform of order μ to eqn. (20), defined by

$$\bar{\bar{w}}_1 (\xi_i) = \int_0^{a} r\bar{w}_1 (r) J_\mu (r\xi_i)\, dr \qquad ...(21)$$

where $J_\mu (r\xi_i)$ denotes the Bessel function of the first kind of order μ and the argument $(r\xi_i)$ and a is a positive root of

$$J_\mu (a\xi_i) = 0. \qquad ...(22)$$

Using the result (Sneddon 1951)

$$\int_0^{a} \left[\frac{d^2 \bar{w}_1}{dr^2} + \frac{1}{r}\frac{d\bar{w}_1}{dr} - \frac{\mu^2}{r^2}\bar{w}_1\right] rJ_\mu (r\,\xi_i)\, dr$$

$$= -\,\xi_i^2\, \bar{\bar{w}}_1 (\xi_i) \qquad ...(23)$$

and the boundary condition (17), we have

$$\bar{\bar{w}}_1(\xi_i) = \frac{(-1)^m \Omega \beta^2 F(\xi_i)}{\mu (\xi_i^2 - \beta^2)} \qquad \ldots(24)$$

where

$$F(\xi_i) = \int_0^a r J_\mu(r\xi_i)\, dr.$$

Applying the inversion formula for the Hankel transform to eqn. (24), we get

$$\bar{w}_1 = \frac{(-1)^m \Omega}{\mu} \frac{\beta_2^2}{a^2} \sum_{i=1}^{\infty} \frac{1}{(\xi_i - \beta^2)} \frac{J_\mu(r\xi_i) F(\xi_i)}{[J'_\mu(a\xi_i)]^2}. \qquad \ldots(25)$$

Using the inversion formula for the cosine transform (Sneddon 1951) to eqn. (25) we have

$$w_1(r, \theta) = \frac{4\Omega\beta^2}{a^2 \alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{\mu} \sum_{i=1}^{\infty} \frac{1}{(\xi_i^2 - \beta^2)} \frac{J(r\xi_i) \cos \mu\theta\, F(\xi_i)}{[J'_\mu(a\xi_i)]^2} \qquad \ldots(26)$$

The relation (14) with the help of (26) gives

$$w_2(r, \theta) = \frac{4\Omega\beta^2}{a^2 \alpha} \frac{1}{1 - \lambda^2 \tau} \sum_{m=0}^{\infty} \frac{(-1)^m}{\mu} \sum_{i=1}^{\infty} \frac{J_\mu(r\xi_i) \cos \mu\theta\, F(\xi_i)}{(\xi_i - \beta^2) [J'_\mu(a\xi_i)]^2}. \qquad \ldots(27)$$

Insertion of w_1 and w_2 in (11) and (12) yields

$$u_3(r, \theta, t) = \frac{4\Omega\beta^2 e^{-\lambda^2 t}}{a^2 \alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{\mu} \sum_{i=1}^{\infty} \frac{J_\mu(r\xi_i) \cos \mu\theta\, F(\xi_i)}{(\xi_i^2 - \beta^2) [J'_\mu(a\xi_i)]^2}. \qquad \ldots(28)$$

$$v_3(r, \theta, t) = \frac{4\Omega\beta^2 e^{-\lambda^2 t}}{a^2 \alpha} \frac{1}{1 - \lambda^2 \tau} \sum_{m=0}^{\infty} \frac{(-1)^m}{\mu} \sum_{i=1}^{\infty} \frac{J_\mu(r\xi_i) \cos \mu\theta\, F(\xi_i)}{(\xi_i^2 - \beta^2) [J'_\mu(a\xi_i)]^2} \qquad \ldots(29)$$

Expressions (28) and (29) express respectively the velocity of the fluid and dust particles.

Since λ^2, τ are positive, the velocity of the dust particles is greater than that of the fluid. When the dust is very fine, the relaxation time of dust particles decreases and ultimately as $\tau \to 0$ the velocities of dust and fluid particle will be same. If the masses of dust particles are small, their influence on the fluid flow is reduced and in the limit as $m \to 0$ the fluid becomes ordinary viscous, and we get the solution of the laminar flow of a viscous fluid through a circular tube whose cross-section is a sector of a circle under the influence of the exponential pressure gradient with respect to time.

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