A NOTE ON SPECIAL-PURPOSE CONVOLUTIONAL CODES

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Two new classes of special-purpose convolutional codes, which correct bursts whose weights lie between certain limits, have been introduced. One class of convolutional codes arises by imposing the weight constraint over the correctable bursts and the other one by more general considerations in which weight constraint has additionally been applied to individual blocks also. The paper also contains lower bounds over the constraint length for these codes.

1. Introduction

Convolutional codes, which were first introduced by Elias (1955) and later systematically treated by Wyner and Ash (1963), have aroused renewed interest because in many practical error control systems they are superior to block codes. These codes are suitable for correcting random as well as burst errors. Wyner and Ash (1963) defined type A, B_1 and B_2 convolutional codes to isolate cases of random and burst error correction. Later Wyner (1963) introduced type C codes. A good account of development of these codes would be found in Peterson and Weldon (1972). In this communication we shall introduce new classes of convolutional codes and obtain lower bounds on the constraint length for them. These arise by generalized considerations of type B_2 and C codes.

A type B_2 convolutional code of block length b will be taken to be capable of correcting bursts of length l or less provided it lies in $r = l/b$ consecutive blocks. And a type C convolutional code will be taken to be capable of correcting any burst error-pattern of length l or less that lies in r consecutive blocks provided that none of the r blocks which may be corrupted have more than e errors ($e \leqslant b$). The entries would be taken from a ground field $GF(q)$.

Type C codes were thought of to economise in constraint length for a situation in which not more than e errors are possible in a block and thereby not more than er errors are possible in a burst of length l which lies in r consecutive blocks. When $e = b$, these reduce to type B_2 codes. Our attention at once goes to the following situations for type B_2 and type C codes :

(i) a type B_2 code correcting bursts of length l or less such that the weight of a correctable pattern satisfies the additional condition that it lies between two numbers w_1 and w_2 because in any communication system, it is natural to expect at least some errors;

(ii) a type C code correcting bursts of length l or less such that while not more than e errors occur in any block, the weight of a correctable pattern further has one of the following conditions :

(a_1) it lies between w_1 and w_2 ($w_1 \leqslant w_2 \leqslant er$) ;

(a_2) no block has less than t errors ($t \leqslant e$), etc.

It is clear that type B_2 and type C codes do not take into account distinction in the above laid down situations and are therefore to be suitably modified to economise in constraint length for meeting these eventualities which might arise in actual practice. We shall be considering slightly more general situations in proposing two new types of convolutional codes in the next two sections. These in particular cover the above mentioned and many more cases. Bounds over the constraint length have been obtained for codes in these new classes.

2. Burst Convolutional Codes of Type C_1

The new classes of convolutional codes arise by considering weight constraint over type C correctable patterns. Earlier, Sharma and Dass (1974) have studied linear codes with weight constraints. First the constraint is kept over the correctable bursts and this leads to type C_1 codes which we define as follows :

Definition—A type C_1 convolutional code with parameter set (l, r; w_1, w_2; e) corrects all bursts of length l or less which lie in r ($= l/b$) consecutive blocks such that the weight of a correctable burst lies between w_1 and w_2 ($w_1 \leqslant w_2 \leqslant er$) and none of the blocks which may be corrupted have more than e errors.

If $w_1 = 0$ and $w_2 = er$, then and then only a type C_1 code reduces to type C code. Thus it would be found that bringing in the concept of weight constraints in terms of w_1 and w_2 gives a rather very general situation in comparison to one covered by type C codes. In what follows, we shall be using the notations followed in Wyner (1963) and Wyner and Ash (1963).

The parity check matrix of the convolutional code would be taken as $A = [B_0, B_m, B_{2m}, \ldots\ldots\ldots]$ where m is a fixed positive integer. By A_N we shall denote the $N \times (N/m)b$ matrix formed by taking the first N rows and first $(N/m)b$ columns of matrix A such that non-zero entries of B_0 are all covered by A_N. Incidentally N is a multiple of m.

In obtaining bounds over N we shall have recourse to a matrix A'_N consisting of the first $N + (r-1)m$ rows and first $N/m + (r-1)b$ columns of A and a $[N + (r-1)m] \times [N + (r-1)m]$ matrix T' given by

$$
T' = \begin{bmatrix}
\left.\begin{matrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{matrix}\right\} m \\
\begin{matrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots 1 \; 0 \cdots & \underbrace{0 \; 0 \; 0}_{m} \end{matrix}
\end{bmatrix}
$$

Definition—A linear combination $X = \sum_{i \in I} a_i C'(i)$ where $a_j \in GF(q)$ and $C'(i)$ is the i-th column of A'_N, will be called a type C_1 correctable combination if $w_1 \leqslant O(I) \leqslant w_2$ where $O(I)$ denotes the order of the set I and not more than e columns of a block of A'_N occur in the summation.

We may state a theorem for type C_1 codes as follows:

Theorem 2.1—If $X = \sum_{i \in I} a_i C'(i)$ and $Y = \sum_{j \in J} b_j C'(j)$ are type C_1 correctable combinations of A'_N then $X = Y$ implies

(1) $I \cap k_l = j \cap k_l$

and

(2) $a_i = b_i \; \forall i \in I \cap k_l$ where $k_l = \{1, 2, \ldots\ldots l\}$

In obtaining the bounds we shall use a result of Wyner which we state in the lemma below:

Lemma (Wyner)—The equivalence relation $\simeq$ on the set of $N + (r-1)m$-vectors, defined by $X \simeq Y$ if $aT'^k(X) = Y$ (or $aT'^k(Y) = X$) for some $a \in GF(q)$ and $k = 1, 2, \ldots N/m + (r-1)$ and the bottom km entries of X (or of Y) being zero, partitions the set of nonzero $N + (r-1)m$-vectors into $q^{N+(r-2)m} \cdot (q^m - 1/q - 1)$ disjoint equivalence classes.

We now prove the following:

Theorem 2.2 (*Lower Bound over N for type C_1 codes*)—For a type C_1 code with parameter set $(l, r; w_1; w_2; e)$,

$$N \geqslant \log_q \alpha - m(r-2) - \log_q \left(\frac{q^m - 1}{q - 1}\right)$$

where

$$\alpha = \Big[\sum_{i=0}^{e} \binom{b}{i}(q-1)^i\Big]^r - \sum_{\sum_p t_p \leqslant w_1 - 1} \Big[\prod_{p=1}^{r} \binom{b}{t_p}(q-1)^{t_p}\Big]$$

$$- \sum_{er \geqslant \sum_p k_p \geqslant w_2+1} \Big[\prod_{p=1}^{r} \binom{b}{k_p}(q-1)^{k_p}\Big]$$

and $0 \leqslant t_p,\ k_p \leqslant e$ for $p = 1, 2, 3, \ldots, r$.

PROOF : For deriving the bound we determine the number of type C_1 correctable combinations from the first r blocks. The number of patterns which have e or less errors in each of the r blocks is

$$\Big[\sum_{i=0}^{e} \binom{b}{i}(q-1)^i\Big]^r . \qquad \ldots(2.1)$$

If there are exactly e_i, $i = 1, 2, \ldots, r$ errors in the i-th block, then the number of combinations having exactly $\sum_{i=1}^{r} e_i$ errors in r blocks is

$$\prod_{i=1}^{r} \Big[\binom{b}{e}(q-1)^{e_i}\Big]. \qquad \ldots(2.2)$$

We have to remove from patterns considered in (2.1), the patterns which are

(i) of weight less than w_1 and

(ii) of weight greater than w_2 and less than er.

The number of patterns of weight less than w_1 is

$$\sum_{\sum t_p \leqslant w_1 - 1} \Big[\prod_{p=1}^{r} \binom{b}{t_p}(q-1)^{t_p}\Big] \qquad \ldots(2.3)$$

and number of patterns of weight greater than w_2 and less than er is

$$\sum_{w_2+1 \leqslant \sum_p k_p \leqslant er} \Big[\prod_{p=1}^{r} \binom{b}{k_p}(q-1)^{k_p}\Big]. \qquad \ldots(2.4)$$

Thus the total number of nonzero type C_1 correctable combinations is α. By Theorem 2.1 and the Lemma, we infer that there should be as many equivalence classes as is the number of type C_1 correctable pattern

Hence

$$q^{N+(r-2)m}\left(\frac{q^m-1}{q-1}\right)\geqslant\alpha.$$

The result follows on taking logarithms to the base q.

Particular Case: If we set $w_1 = 1$ and $w_2 = er$, the idea of weight of bursts becomes redundant and no choice of k_p's is possible, satisfying

$$er \geqslant \sum_p k_p \geqslant er + 1.$$

Hence (2.4) reduces to zero.

Also with $w_1 = 1$, (2.3) reduces to unity.

Hence $\alpha = \left[\sum_{i=0}^{e}\binom{b}{i}(q-1)^i\right]^r - 1$

and the bound reduces to one obtained by Wyner (1963) for type C codes

3. Burst Convolutional Codes of Type C_2

In the class C_1 of convolutional codes, the burst weight was taken to be lying between w_1 and w_2 but in blocks there could be any number of e or less errors. It is quite natural to think that even each block would have at least a certain number of errors not exceeding e. The type C_2 codes are proposed to cover this eventuality, the additional weight constraints now being over the blocks as well. The new class of codes that arises would be called type C_2 convolutional codes, which we define as follows:

Definition—A type C_2 burst correcting convolutional code with parameter set $(l, r; w_1, w_2; t, e)$ corrects bursts of length l or less whose weight lies between w_1 and w_2 and each of the r blocks which may be corrupted have at least t and at the most e errors.

If $t = 0$, then a type C_2 code reduces to type C_1 code discussed in the previous section. A general situation of logically extreme type is obtained by considering that the i-th block has at least t_i and at most e_i $(t_i \leqslant e_i \leqslant b)$ errors, $i = 1, 2, \ldots, r$. However we shall consider only the case $t_1 = t_2 = \ldots = t_r = t$ and $e_1 = e_2 = \ldots e_r = e$, which is intended in the above definition.

Definition—A linear combination $X = \sum_{i \in I} a_i C'(i)$ of columns of A'_N is called a type C_2 correctable combination is $w_1 \leqslant 0(I) \leqslant w_2$ and the number of $C'(i)$'s from any one block which appear in X, lies between t and e.

Theorem 3.1—For a type C_2 code with parameter set $(l, r; w_1, w_2; t, e)$

$$N \geqslant \log_q \beta - m(r-2) - \log_q\left(\frac{q^m-1}{q-1}\right)$$

where

$$\beta = \left[\sum_{i=t}^{e} \binom{b}{i}(q-1)^i\right]^r - \sum_{tr \leqslant \sum_p k_p \leqslant w_1 - 1} \left[\prod_{p=1}^{r} \binom{b}{k_p}(q-1)^{k_p}\right]$$

$$- \sum_{w_2+1 \leqslant \sum_p k_p \leqslant er} \left[\prod_{p=1}^{r} \binom{b}{k_p}(q-1)^{k_p}\right]$$

such that $t \leqslant k_p \leqslant e$ for $p = 1, 2, \ldots, r$.

PROOF : The number of all possible combinations having at least t and at the most e errors in each of the r blocks is

$$\left[\sum_{i=t}^{e} \binom{b}{i}(q-1)^i\right]^r . \qquad \ldots(3.1)$$

Out of these combinations

$$\sum_{tr \leqslant \sum_p k_p \leqslant w_1 - 1} \left[\prod_{p=1}^{r} \binom{b}{k_p}(q-1)^{k_p}\right] \qquad \ldots(3.2)$$

combinations have weight between tr and $w_1 - 1$ and

$$\sum_{w_2+1 \leqslant \sum_p k_p \leqslant er} \left[\prod_{p=1}^{r} \binom{b}{k_p}(q-1)^{k_p}\right] \qquad \ldots(3.3)$$

have between $w_2 + 1$ and er.

Hence total number of distinct type C_2 correctable linear combinations is β indeed. As these combinations should be in different equivalence classes, we get

$$N \geqslant \log_q \beta - m(r-2) - \log_q \left(\frac{q^m - 1}{q - 1}\right).$$

Next we give an example of a type C_2 code.

Example: Consider a convolutional code for which

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

This defines a type C_2 code with $m = 1$, $b = 3$, $N = 4$, $l = 3$, $r = l/b = 1$, $t = 2$ and $e = 3$. This can be seen by considering its A'_N matrix given by

$$A'_N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Remark : Viterbi (1967) gave a decoding algorithm which is particularly effective in decoding convolutional codes of short constraint length. So it is sometimes useful to have economy in constraint length.

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