ON QU-OPERATORS

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An operator $T \in B(H)$ is said to be quasi-unitary (QU—for short) if and only if $TT^* = T^*T = T + T^*$. Properties of QU-operators regarding norm, spectrum properties, similarity and unitary equivalence are studied. We show further that for a QU-operator T: $(i)(-T)$ is an accretive operator and (ii) Cayley transform of T is a contraction. In addition we also study condition for what operators the product of QU-operators with them is a QU-operator and such properties.

1. INTRODUCTION

Paschke (1972) has considered the concept of U*-algebra due to Palmer (1971) in terms of quasi-unitary elements. According to him, x in A, a Banach *-algebra, is quasi-unitary if and only if $xx^* = x^*x = x + x^*$. This suggests that the study of an operator on Hilbert space H satisfying similar conditions will be of interest.

Definition—An operator T on a Hilbert space will be called quasi-unitary (QU-operator, for short) if

$$TT^* = T^*T = T + T^*.$$

Note that by an operator we mean the elements of the Banach *-algebra $B(H)$ of all bounded linear transformations on a Hilbert space H.

West (1965) proves the following result:

" If $T \in B(H)$ is compact and normal, then $\sigma(T) \subset C_1$, the unit circle with centre at 1 iff $TT^* = T + T^*$. " This also suggests that the study of QU-operators is worth undertaking.

In the course of our investigations we have obtained various properties of QU-operators such as norm properties, spectral properties, similarity and unitary equivalence. In addition certain conditions implying quasi-uniticity are exhibited and some results regarding the product of an operator with a QU-operator are established. We have characterized a QU-operator T in terms of $T - I$. Rudin (1974; prob. 13, p. 324) states the problem: " If $T \in B(H)$ is normal, show that $T^* = UT$ for some unitary U. When U is unique ? " Our discussion here shows that for a QU-operator T, the operator U turns out to be unique and it is equal to $(T^* - I)$.

Here, we furnish some non-trivial examples of QU-operators:

1. The most simple example will be the operator $T = 2I$. It should be noted that $T = I$ does not serve as an example of a QU-operator.

2. $T = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

3. $T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

4. $T = \begin{bmatrix} 0 & 0 \\ 0 & 1 - i \end{bmatrix}$.

2. Properties of QU-Operators

In this section we state the basic properties of a QU-operator. The proofs are omitted as they are straightforward.

Properties

2.1. Let T_1, $T_2 \in B(H)$ be QU. Then $T_1 + T_2$ is QU iff $T_1 T_2^* + T_2 T_1^* = T_1^* T_2 + T_2^* T_1 = 0$.

2.2. The class of QU-operators forms a closed subset of $B(H)$.

2.3. Let $T \in B(H)$ be QU; which is also a contraction. Then

$$T^* = - \sum_{k=1}^{\infty} T^k.$$

2.4. Let $T \in B(H)$ be QU and $S \in B(H)$ be such that $\| (I - T) - S \| < 1$, then S is invertible. (See Helmberg 1969, p. 162).

2.5. Let $T \in B(H)$ be QU. Then $(T - I)T^*T = T^2 = T^*T(T - I)$.

Lemma 2.1—Let $T \in B(H)$ be QU. Then

$$\| Tx + x \|^2 = 2 \| Tx \|^2 + \| x \|^2 \text{ for } x \in H.$$

Result 2.1—Let $T \in B(H)$ be QU. Then $R(T + I)$, the range of $(T + I)$ is closed if and only if the graph T is closed.

Proof: By Lemma 2.1, we have

$$\| Tx + x \|^2 = 2 \| Tx \|^2 + \| x \|^2, \qquad (x \in H).$$

We can, argue on the lines of Rudin (1974), that the mapping $(T + I)x \leftrightarrow (x, \sqrt{2}\, Tx)$ is an isometric one-one correspondence. Hence $R(T + I)$ is closed if and only if the graph of $\sqrt{2}\, T$ is closed, and from this the result follows.

Result 2.2—Let $T \in B(H)$ be *QU*. Then $T + I$ is an isomorphism on H.

Proof: Suppose $(T + I)x = (T + I)y$. Then $(T + I)$ $(x - y) = 0$. So that by Lemma 2.1, we have $\| x - y \|^2 = 0$, i.e. $x = y$. Hence $T + I$ is an isomorphism on H.

To complete this section we state some norm properties of a QU-operator.

Result 2.3—Let $T \in B(H)$ be QU. Then

(1) $\| T \| \leqslant 2$.

(2) $\| T \|^2 = \| T^2 \| = \| T + T^* \|$.

(3) $\| T^* Tx \| - \| Tx \| \leqslant \frac{1}{2} \| T^2 \| \| x \|$ for $x \in H$.

(4) $\| T^2 x \| = \sqrt{2} [\| Tx \|^2 + \operatorname{Re}(T^2 x, x)]^{1/2}$.

(5) $\| Tx \|^2 \leqslant \| x \| [\| Tx \| + 2 \| x \|]$ for $x \in H$.

3. Conditions Implying QU-ness

Result 3.1—$T \in B(H)$ is a QU-operator if and only if $(T - I)$ is a unitary operator.

Result 3.2—Let $T \in B(H)$ be an operator with S and R as its real and imaginary parts respectively. Then T is QU iff $SR = RS$ and $(S - I)^2 = (I - R)(I + R)$.

Result 3.3—Let $T \in B(H)$ be hyponormal such that $\sigma(T - I)$ lies on the unit circle. Then T is QU.

Result 3.4—Let $T \in B(H)$ be QU and unitary then T^{-1} is also QU.

The proofs of the above results are straightforward hence are omitted.

Result 3.5—Let $T \in B(H)$ be such that

$$R = (T^*T)^{1/2}, \qquad S = (TT^*)^{1/2} \quad \text{and} \quad M = \operatorname{Re}(T).$$

If each of R and S commutes with M and $R^2 + S^2 - 4M = 0$ then T is a QU-operator.

Proof: It follows by Embry's result, viz. " If each of R and S commutes with M, then T is normal " (Embry 1966).

Remarks

3.1. Since $T + I$ and $T - I$ are one-one, ± 1 do not belong to $\sigma(T)$ (Riesz and Nagy 1955).

3.2. Let $T - I$ be a unimodular contraction on H. Then $\sigma(T - I) = \phi$ and eigenspaces corresponding to distinct eigenvalues of $T - I$ are orthogonal. If H is spanned by eigenvectors of $(T - I)$, then T is a QU-operator (Russo 1968, Saito 1973).

3.3. Let $T - I$ be a unimodular contraction. Then $\sigma(T - I)$ is a spectral set for $T - I$ iff T is a QU-operator (Russo 1968, Saito 1973).

4. Similarity and Unitarily Equivalence

Result 4.1—If T, $S \in B(H)$, T is QU and S is unitarily equivalent to T then S is also a QU-operator.

Proof: Since T and S are unitarily equivalent $S = U^*TU$, where U is unitary. Consider,

$$SS^* = (U^*TU)(U^*TU)^* = U^*TT^*U.$$

Similarly each of S^*S and $S + S^*$ comes out to be U^*TT^*U and hence S is QU.

Result 4.2—If M and N are invertible operators and T is QU with M and N similar with respect to $(T-I)$ then so are $(N^*N)^{-1}$ and $(M^*M)^{-1}$.

Proof: Since M and N are similar with respect to $T-I$, we have $M = (T-I)\,N(T-I)^{-1}$. But $T-I$ is unitary so that $(T-I)^{-1} = T^*-I$. Hence we have

$$(T^*-I) = N(T^*-I)\,M^{-1}.$$

This implies that

$$(T-I) = M^{*-1}(T-I)\,N^*. \qquad \ldots(1)$$

Also

$$M(T-I) = (T-I)\,N.$$

Hence

$$(T-I) = M(T-I)\,N^{-1}. \qquad \ldots(2)$$

From (1) and (2) we get

$$M(T-I)\,N^{-1} = M^{*-1}(T-I)\,N^*.$$

Hence

$$M(T-1)\,N^{-1}N^{*-1} = M^{*-1}(T-I)$$

which gives

$$(T-I)(N^*N)^{-1}(T-I)^{-1} = (M^*M)^{-1}$$

which shows that $(N^*N)^{-1}$ and $(M^*M)^{-1}$ are similar with respect to $(T-I)$.

Remarks: If T, M, $N \in B(H)$ are such that, N is invertible, T is QU and M and N are similar with respect to $(T-I)$ then so are M and M^*N^{*-1} N if M^* commutes with $(T-I)$.

5. Spectral Properties of QU-operators

Result 5.1—If α, β are two eigenvalues of T corresponding to two eigenvectors x and y then

$$x \perp y \text{ or } \bar{\beta} = \frac{2\,\mathrm{Re}(\alpha)}{\alpha}.$$

PROOF : Consider

$$\begin{aligned}\alpha\bar{\beta}(x, y) &= (\alpha x, \beta y) = (Tx, Ty)\\ &= (T^*Tx, y) = ((T + T^*)x, y)\\ &= (Tx, y) + (T^*x, y)\\ &= (\alpha x, y) + (\bar{\alpha}x, y)\\ &= (\alpha + \bar{\alpha})(x, y).\end{aligned}$$

Thus

$$\{\alpha\bar{\beta} - (\alpha + \bar{\alpha})\}(x, y) = 0.$$

Thus we have either $(x, y) = 0$ or $\alpha\bar{\beta} = \alpha + \bar{\alpha} = 2\ \mathrm{Re}(\alpha)$. From this we conclude that $x \perp y$ or $\bar{\beta} = \dfrac{2\,\mathrm{Re}(\alpha)}{\alpha}$.

Remark: If $\beta = \alpha$, then $\bar{\alpha} = \dfrac{\alpha}{\alpha - 1}$.

α will be real if $\alpha = 0$ or $\alpha = 2$.

Similar result holds if α is replaced by β.

Result 5.2—If T is a QU-operator which is invertible, then

$$\sigma(T^*) = \sigma(I + T^{-1}T^*)$$

and

$$\sigma(T) = \sigma(I + T^{*-1}T).$$

Remark: For a *QU*-operator T,

$$0 \in \sigma(T) \text{ if and only if } 0 \in \sigma\ (T + T^*).$$

Result 5.3—If T is a QU-operator, then

(1) $\omega(T + T^*) = \rho(T + T^*) = \omega(T^2)$

(2) $\omega(T^2) \leqslant 2\omega(T) = 2\rho(T)$.

PROOF: Since T is QU, it is normal. Hence $\omega(T) = \rho(T) = \| T \|$ (Halmos 1967).

Result 5.4—For a QU-operator T on H, $\mathrm{Re}(T) \geqslant 0$, and $\{\lambda\colon \mathrm{Re}\,\lambda \geqslant 0\}$ is a spectral set for T.

PROOF: It is obvious that for any operator T, we have

$$\| (T + I)x \|^2 - \| (T - I)x \|^2 = 4\,\mathrm{Re}(Tx, x).$$

In addition, T is QU-operator and hence on account of Lemma 2.1 we obtain

$$\| (T + I)x \|^2 - \| (T - I)x \|^2 = 2\,\| Tx \|^2$$

Thus, $4\,\mathrm{Re}\,(Tx, x) = 2\,\| Tx \|^2$ which shows that $\mathrm{Re}\,(Tx, x) \geqslant 0$. That is $\mathrm{Re}\,(T) \geqslant 0$. Hence $(T + I)^{-1}$ exists and $\| (T - 1)(T + I)^{-1} \| \leqslant 1$. This shows that $\{\lambda: \mathrm{Re}\,\lambda \geqslant 0\}$ is a spectral set for T (Riesz and Nagy 1955).

Remark: The above result leads us to the conclusion that if T is a QU-operator on H, then $-T$ is an accretive operator (Fillmore 1968). In fact much more is true.

Result 5.5—Let T be a QU-operator on H. Then $(T + I)^{-1}$ is a contraction and Cayley transform of T exists which is a contraction.

PROOF: We have seen in the proof of result (5.4), that $\| (T - I)(T + I)^{-1} \| \leqslant 1$. But if T is QU-operator then we know that $(T - I)$ is unitary and hence $\| (T + I)^{-1} \| \leqslant 1$; that is $(T + I)^{-1}$ is a contraction.

We can now consider $S = (T - I)(T + I)^{-1}$ as the Cayley transform for T, which is a contraction as was seen earlier (Fillmore 1968).

Result 5.6—Let T be a QU-operator on H, then

$$\sigma(T) = \{ \lambda \in \mathbb{C}: \ \lambda = \mu + 1, \quad | \mu | = 1 \}$$

where $\mu \in \mathbb{C}$.

PROOF: Since T is a QU-operator, $(T - I)$ is unitary. Hence $\sigma(T - I)$ lies on the unit circle.

Further,

$$\sigma(T) = \sigma(T - I + I) \subseteq \sigma(T - I) + \{1\}.$$

$$\subseteq \{\mu \in \mathbb{C}: \ | \mu | = 1\} + \{1\}.$$

Thus if $\lambda \in \sigma(T)$, then $\lambda = \mu + 1$ where $| \mu | = 1$.

Result 5.7.—If T is a QU-operator on H, and μ is a complex number such that $| \mu - 1 | < 1$ then μ is a regular value.

PROOF: We have the result from Helmberg (1969): "If λ is regular value of an operator A and

$$| \mu - \lambda | < \frac{1}{\| (A - \lambda I)^{-1} \|},$$

then μ is also regular value."

It is clear that 1 is a regular value for a QU-operator T. Hence $| \mu - 1 | < 1$ implies that

$$| \mu - 1 | < \frac{1}{\| T - I \|} = \frac{1}{\| (T - I)^{-1} \|},$$

and thus μ is a regular value.

Remark: As a consequence of result (5.7) and $\| T \| \leqslant 2$ we see that for a QU-operator T,

$$\sigma(T) \subset \{z \in \mathbb{C}: \ | z | \leqslant 2\} - \{z \in \mathbb{C}: \ | z - 1 | < 1 \}.$$

Result 5.8—If T is a QU-operator on H and z is any point in the complex plane, then

$$d(z, W(T)) \leqslant 1 + |z - 1|.$$

Here $d(z, W(T))$ is the distance of z from $W(T)$.

PROOF: We shall use the following result from Stampfli (1965):

" For any operator $T \in B(H)$, we have

$$\|(T - zI)x\| \geqslant d(z, W(T)) \quad \text{where} \quad \|x\| = 1."$$

Now,

$$\|(T - zI)x\| = \|(T - I + I - zI)x\|$$
$$\leqslant \|(T - I)x\| \; \|(I - zI)x\|$$
$$= \|x\| + |1 - z| \, \|x\|.$$

Hence the result follows.

6. On Product of Operator with a QU-Operator

Result 6.1—Let T be a non-zero QU-operator on H. Then there exists a self-adjoint operator S on H such that $ST = TS$ and $ST^* = T^*S$ and in this case

$$T^* = T[e^{-i\pi/2} S + T - I].$$

PROOF: Since T is given by $S_0 = T - T^*$, let $T_0 = e^{-i\pi/2} T$ and $S = e^{-i\pi/2} S_0$, then $S = T_0 + T_0^*$ and this is clearly self-adjoint. Now consider,

$$ST = (T_0 + T_0^*)T = (e^{-i\pi/2} T + e^{i\pi/2} T^*)T$$
$$= e^{-i\pi/2} T^2 + e^{i\pi/2} T^*T.$$

Similarly $TS = e^{-i\pi/2} T^2 + e^{i\pi/2} TT^*$. The quasi-unitary nature of the operator T gives us $TS = ST$. Also

$$ST^* = (T_0 + T_0^*)T^* = (e^{-i\pi/2} T + e^{i\pi/2} T^*)T^*$$
$$= e^{-i\pi/2} TT^* + e^{i\pi/2} T^{*2}$$

and

$$T^*S = e^{i\pi/2} T^{*2} + e^{-i\pi/2} T^*T.$$

This leads to the conclusion that $ST^* = T^*S$.

Next

$$TS = e^{-i\pi/2} T^2 + e^{i\pi/2} TT^*$$
$$= e^{-i\pi/2} T^2 + e^{i\pi/2}(T + T^*).$$

Hence,

$e^{i\pi/2} T^* = TS - e^{i\pi/2} T - e^{-i\pi/2} T^2$ and we get the remaining part.

Result 6.2—Let T be a QU-operator on H and $ST = TS$ for any $S \in B(H)$. Then ST will be a *QU*-operator on H if S is a projection on H.

PROOF: Suppose S to be a projection on H in the sense that $S = S^*$ and $S^2 = S$. Consider

$$(ST)(ST)^* = (ST)(T^*S^*) = STS^*T^* = STST^*$$
$$= S^2TT^* = STT^*$$

and

$$(ST) + (ST)^* = ST + T^*S^* = ST + ST^*$$
$$= S(T + T^*) = STT^*.$$

Similarly we have $(ST)^* \ (ST) = T^*SST = ST^*ST = SST^*T = ST^*T$ (Rudin 1974).

Thus the proof of the theorem is complete.

Result 6.3—If T is a QU-operator on H, $ST = TS$ for a self-adjoint operator S on H, then $(S + T)$ is a QU-operator on H if and only if $2S = S(TT^* + S)$.

Remarks—If in addition S is idempotent operator on H, then $(S + T)$ will be a QU-operator on H if and only if $S = STT^*$.

Result 6.4—Let $T \in B(H)$ be an isometry such that either $I - TT^*$ or $I + TT^*$ is a QU-operator. Then T is unitary.

PROOF: Since $I - TT^*$ is QU, we have

$$(I - TT^*)^2 = 2I - 2TT^*.$$

Also

$$(I - TT^*)^2 = I - TT^*.$$

Thus $I - TT^* = 0$ implying $TT^* = I$.

Similar arguments will give the proof regarding $I + TT^*$. Hence the result follows.

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