COMPRESSIBILITY INFLUENCE ON INTERNAL WAVES EXCITED BY TRAVELLING FORCING EFFECTS

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Compressibility effects on two-dimensional wave patterns produced in a density-stratified atmosphere by forcing effects travelling with uniform velocity in an arbitrary direction are studied using Lighthill's technique. A steady forcing effect moving horizontally excites unattenuated waves travelling both ahead and behind in a 'column' parallel to the forcing line. A non-zero frequency parameter f_0 generates wave systems travelling downstream with a multi-cusped wave crests while splitting certain steady case wave systems into two, which are otherwise coincident. When the compressibility effects are minimal the forcing effect associated with $f_0 \geqslant 2$ excites acoustic waves in all directions while the wavelength of these waves becomes smaller and smaller as f_0 increases. When the compressibility effects are strong, the waves always trail behind the forcing region for any value of f_0 including zero.

1. INTRODUCTION

In upper atmosphere or inside the Earth's core many types of wave systems are possible. It is desirable to know how they are excited by different types of travelling forcing effects. With this motivation, the author (see Sarma and Sarma 1973), following a technique developed by Lighthill (1960), discussed waves generated inside the Earth's core to explain the westward drift of the geomagnetic field. Rarity (1967), Stevenson (1969), Stevenson and Thomas (1969) and others studied internal waves in incompressible, inhomogeneous fluids. In any study of atmospheric waves it is important to determine the effects of compressibility. Recently, Rarity (1973) has accounted for such effects on two-dimensional internal waves excited by steady forcing effects. More recently, Peat and Stevenson (1975) considered internal waves and Cauchy-Poisson waves excited by impulsively started bodies in an isothermal atmosphere. It is the purpose of the present analysis to find compressibility influence on internal waves excited by disturbances of varying intensity in different directions in the atmosphere. This study is of meteorological interest since the atmospheric winds change often in direction and intensity.

We first consider minimal compressibility (Mach number $M^2 = 0 \cdot 1$) effects in § 3 where it is shown that a steady forcing effect moving horizontally produces nearly semi-circular waves behind the forcing region propagating in an arbitrary direction. In addition, unattenuated disturbances form a 'column' upstream and downstream. An oscillatory forcing effect excites waves which trail behind with four-cusped wave crests forming caustics along the forcing line. When the frequency of oscillation exceeds certain value, the waves are excited all around the forcing region. When compressibility is strong ($M^2 = 2$), a steady forcing effect moving horizontally generates a 'column' of waves upstream and downstream just as in the case of minimal compressibility. However, it is interesting to note that the periodic nature of the forcing effect of any frequency is not capable of producing forward influence. Clearly, this is in contrast with the minimal compressibility case. The waves trail behind with tri-cusped wave crests forming caustics along the forcing line. We also find wave crests which are bow-shaped either concave upwards or downwards. These types of waves are characteristic of sound waves and are split into two systems which are otherwise coincident in the steady case.

2. Dispersion Relation

It can be easily shown that the perturbed density ρ^1 in a two-dimensional compressible, density-stratified fluid flow satisfies the following differential equation (see Rarity 1973 for all details):

$$\frac{\partial^4 \rho'}{\partial t^4} - a_0^2 \frac{N^2}{g} \frac{\partial^3 \rho'}{\partial y \partial t^2} - a_0^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial^2 \rho'}{\partial t^2} - a_0^2 N^{*2} \frac{\partial^2 \rho'}{\partial x^2} = 0, \qquad \ldots(1)$$

where

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho_0}{dy}, \quad N^{*2} = N^2 - g^2/a_0^2, \quad a_0^2 = \gamma p_0/\rho_0,$$

γ is the ratio of specific heats, p_0 and ρ_0 are unperturbed pressure and density respectively; x, y are the spatial coordinates and t is time. The effects of the forcing region can be incorporated in the governing differential equation (1) by replacing the right-hand side by a non-zero forcing term

$$e^{-i\sigma_0 t} f(\bar{r} - \bar{U}t) \qquad \ldots(2)$$

where $\bar{r} = x\bar{i} + y\bar{j}$ is the position vector and $\bar{U} = U \cos \alpha \bar{i} + U \sin \alpha j$. The differential equation admits a plane wave solution of the type

$$\rho' \propto \exp\left[i(-\sigma t + lx + my)\right] \qquad \ldots(3)$$

if the dispersion relation

$$S(\sigma_0, l, m) = (\sigma_0 + Ul \cos \alpha + Um \sin \alpha)^4 + im \frac{N^2}{g} a_0^2 \sigma^2 - a_0^2 (\sigma_0 + Ul \cos \alpha + Um \sin \alpha)^2 (l^2 + m^2) + a_0^2 N^{*2} l^2 = 0 \qquad \ldots(4)$$

is satisfied ($\sigma = \sigma_0 + Ul \cos\alpha + Um \sin\alpha$ due to Doppler effect). By taking Fourier transforms a formal solution of (1) with (2) as its right-hand side may be obtained as

$$\rho^1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F(\bar{k}) \exp [i\{-\sigma_0 t + \bar{k}\}.(\bar{r} - \bar{U}t)\}]}{S(\sigma_0 + \bar{U}.\bar{k}, l, m)} \, dl \, dm \qquad ...(5)$$

where

$$f(\bar{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\bar{k}) \, e^{i\bar{k}\cdot\bar{r}} \, dl \, dm \qquad ...(6)$$

and $\bar{k} = l\bar{i} + m\bar{j}$ is the wave-number vector. We introduce the transformations

$$n = m - i\frac{N^2}{2g}$$

$$N^* n_1 = Ul \cos\alpha + Un \sin\alpha$$

$$N^* l_1 = Ul \sin\alpha - Un \cos\alpha$$

$$Ux_1 = N^*(x \sin\alpha - y \cos\alpha)$$

$$Uy_1 = N^*(x \cos\alpha + y \sin\alpha) \qquad ...(7)$$

and also notice that $N^{*2} = N^2(\gamma - 1)/\gamma$ and $N^2 = \gamma g^2/a_0^2$ as a result of the assumption that the speed of sound is independent of y (see Rarity 1973). Further, we take $\gamma = 7/5$ [and hence $\gamma^2/4(\gamma - 1)$ is $1\cdot225$] which is the value for a diatomic gas. Then the dispersion relation (4) becomes

$$S_1(f_0, l_1, n_1) = M^2(f_0 + n_1)^4 - (f_0 + n_1)^2(l_1^2 + n_1^2 + \beta) + (l_1 \sin\alpha + n_1 \cos\alpha)^2 = 0 \qquad ...(8)$$

where

$$M^2 = U^2/a_0^2, \quad f_0 = \sigma_0/N^* \quad \text{and} \quad \beta = 1\cdot225\, M^2.$$

A method of obtaining a unique solution (satisfying a radiation condition) of integral equation (5) is explained in Lighthill (1960, 1967). In wave-number (l_1, n_1) space at each point on $S_1(f_0) = 0$ we draw an arrow normal to the curve, choosing from the two normal directions the one pointing towards the curve $S_1(f_0 + \delta) = 0$ with δ positive and small. The amplitude of the waves generated by the forcing term is asymptotically given by

$$\left[\frac{(2\pi)^{3/2}}{R^{1/2}\,|\kappa|^{1/2}}\right]\left[\frac{F(\bar{k}_1)}{\nabla S_1(f_0)}\right] \qquad ...(9)$$

where R is the distance from the forcing region, $\bar{k}_1$ is the wave-number vector in (l_1, n_1) space, ∇ is the operator grade with respect to (l_1, n_1) space and κ is the curvature of the wave-number curve. A straight portion of the wave-number curve generates waves without attenuation, the first factor in (9) being replaced by 2π. We obtain the curves of constant phase using the formula

$$\bar{r}_1 = \phi^2 \frac{\nabla S_1}{|\bar{k}_1 \cdot \nabla S_1|} \operatorname{Sgn}\left(-\frac{\partial S_1}{\partial f_0}\right), \qquad \ldots(10)$$

where $\bar{k}_1 \cdot \bar{r}_1 = \phi$.

3. Description of Wave Pattern for Minimal Compressibility

It is clear that the effects of compressibility are determined by the magnitude of M^2 relative to unity. We choose two cases of interest, namely $M^2 = 0 \cdot 1$ (when the effects are minimal) and $M^2 = 2$ (when the effects are strong). The frequency spectrum of $f_0 \neq 0$ is divided into two regimes so that in any regime S_1 has the same features for all f_0. The first regime is $0 < f_0 < 2$ (hereafter called R_1) and the second one is $f_0 \geqslant 2$ (hereafter called R_2). Two values of $f_0 = 0 \cdot 5$ and 2 are chosen to represent f_0 in these two regimes. Moreover, we study forcing effects moving horizontally and those moving in a direction $\alpha = 30°$.

3.1. *Horizontally Moving Forcing Effects*

For $\alpha = 0$ the dispersion relation (8) takes the form

$$S_1(f_0, l_1, n_1) = M^2(f_0 + n_1)^4 - (f_0 + n_1)^2(l_1^2 + n_1^2 + \beta) + n_1^2 = 0. \qquad \ldots(11)$$

For a steady forcing effect, we get

$$S_1(o_1, l_1, n_1) = n_1^2[M^2 n_1^2 - l_1^2 - n_1^2 - \beta + 1] = 0 \qquad \ldots(12)$$

which splits into

$$M^2 n_1^2 - l_1^2 - n_1^2 - \beta + 1 = 0 \qquad \ldots(13)$$

and

$$n_1^2 = 0. \qquad \ldots(14)$$

The part of the curve (14) is a straight portion consisting of the line $n_1 = 0$ taken twice [see Fig. 1 (*a*)]. The arrows along the appropriate normal must be drawn on the two straight lines and the normal directions appropriate at each line may or may not coincide. When ϵ is positive but very small, the double pole at $n_1 = 0$ splits into simple poles at

$$n_{11} = \frac{-i\epsilon(l_1^2 + \beta)^{\frac{1}{2}}}{(l_1^2 + \beta)^{1/2} + 1}, \qquad n_{12} = \frac{-i\epsilon(l_1^2 + \beta)^{\frac{1}{2}}}{(l_1^2 + \beta)^{1/2} - 1} \qquad \ldots(15)$$

The reader is referred to Lighthill (1967) for the method of obtaining (15). The waves associated with the straight portion (14) are one-dimensional waves which do not attenuate. These waves travel upstream if the transverse wave-numbers satisfy

$$l_1^2 + \beta - 1 < 0. \qquad \ldots(16)$$

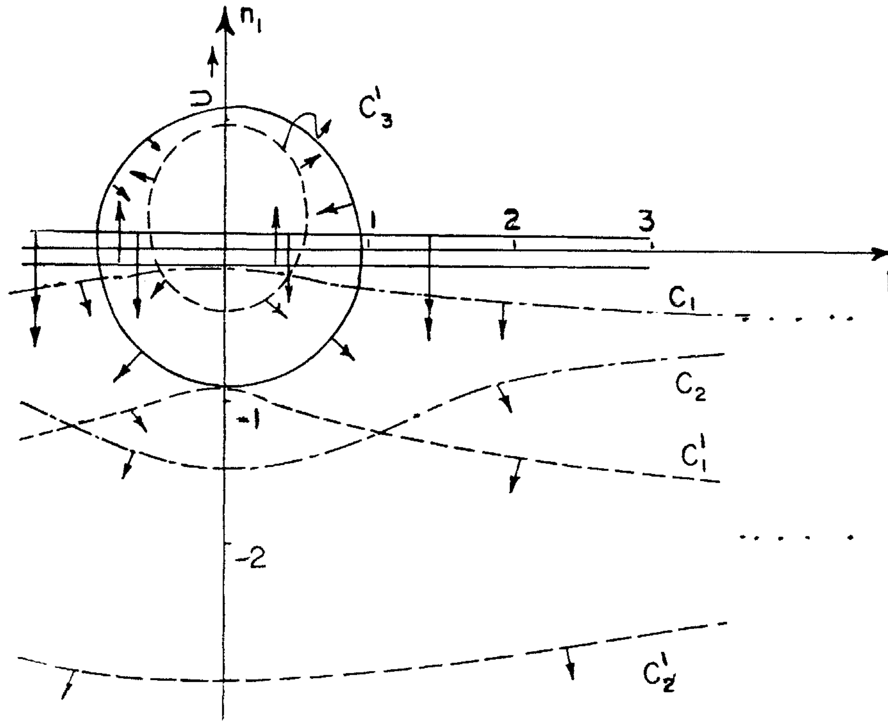

FIG. 1 (*a*). Wave-number curves S_1 for $M^2 = 0{\cdot}1$, $\alpha = 0$, —— $f_0 = 0$, — — — $f_0 = 0{\cdot}5$, ——— $f_0 = 2$, ... asymptotes.

These disturbances are created by the forcing region modified by a 'low-pass filter' passing wave-numbers below $(1 - \beta)^{1/2}$. The disturbances extending downstream are not subjected merely to the complimentary 'high-pass filter'; they include some low wave-numbers also. Each of these waves originate at the forcing effect and thereafter, being independent of y, propagate either upstream or downstream. After a long enough time they extend far up and down forming a 'column' of waves ahead and behind the forcing region. It is known that such a column (called Taylor Column) is generated by steady forcing effects in incompressible rotating fluids (Lighthill 1967) and in electrically conducting, rotating, incompressible fluids (Sarma and Sarma 1973). Here we find the 'column' excited in a compressible stratified fluids in minimal or strong compressibility cases [see Figs. 1 (*a*), 3 (*a*)].

The wave-number curve corresponding to (13) is plotted in Fig. 1 (*a*) which is nearly a circle. The waves corresponding to the points on it propagate in an arbitrary direction. These waves trail behind the forcing region first filling the fluid region behind it.

The wave-number curves for the regime R_1 are shown again in Fig. 1 (*a*). The waves corresponding to C_1 and C_2 [see Fig. 1 (*a*)] propagate downstream with wave crests [see Fig. 1 (*b*)] cusped at four points looking like an inverted funnel. The caustics along the forcing line indicate an accumulation of wave

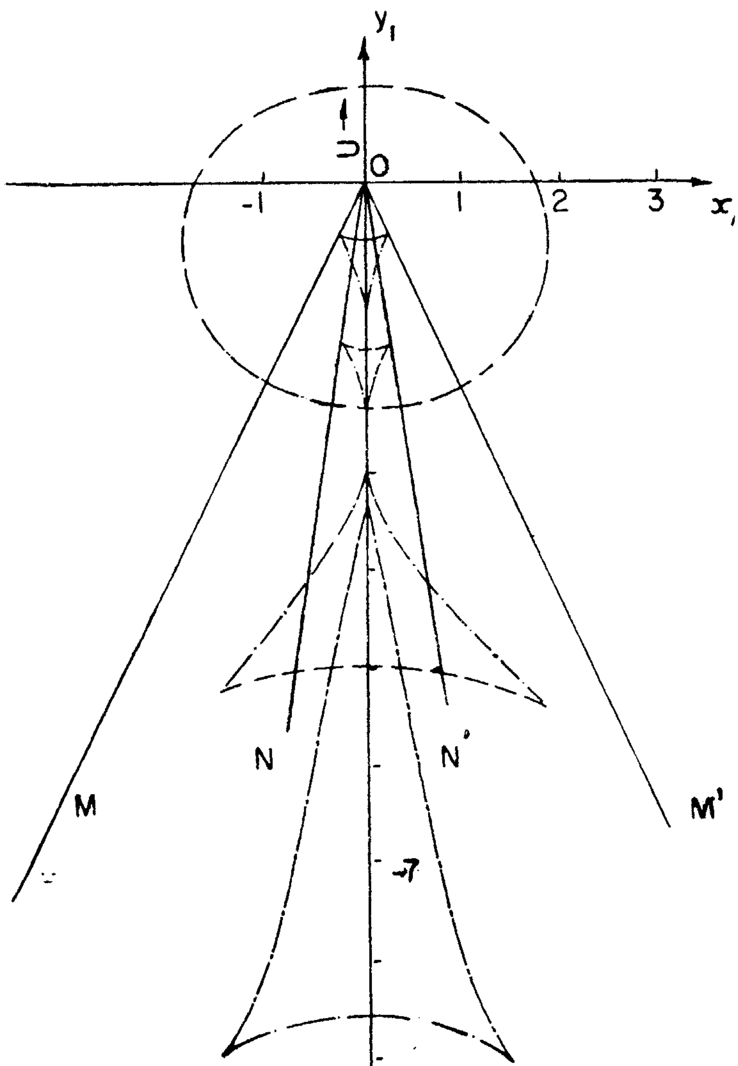

FIG. 1 (*b*). Curves of constant phase corresponding to Fig. 1 (*a*). The notation for f_0 is the same as in Fig. 1 (*a*) in this figure and subsequent figures.

crests. The waves are confined to the wedge (OM, OM') formed by the loci of the two of the four cusps.

The wave-number curves are also drawn for the regime R_2 in Fig. 1 (*a*). In this case, the waves corresponding to the wave-number curves C_1', C_2' form a system of waves similar to the system in the regime R_1. However, the waves are not confined to the wedge formed by (ON, ON'). The waves of longer wavelength travel outside this wedge and those waves of smaller wavelength travel inside the wedge. There is another system of waves associated with points on C_8'.

The waves of this system propagate all around the forcing region. As the frequency increases, the waves of smaller and smaller wavelength are generated in this system. This system is characteristic of acoustic waves.

3.2. *Forcing Effect in the Direction of* $\alpha = 30°$

In the steady case, the waves propagate downstream with wave crests accumulated on the left side of the forcing line [see Fig. 2 (*b*)]. Also the waves propagate downstream and upstream on the right side of the forcing line disturbing the whole atmospheric region.

In the regime R_1, there is a single system of waves propagating behind the forcing effect with tripple-cusped wave crests associated with points on C_1 and C_2 [see Fig. 2 (*b*)]. In the regime R_2, there is one system similar to the above

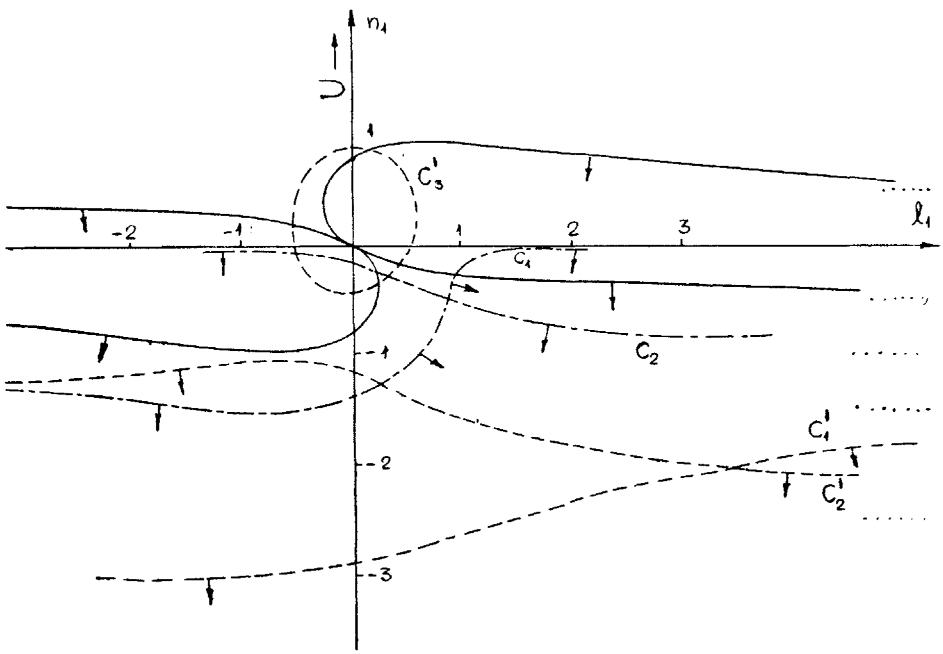

FIG. 2 (*a*). Wave-number curves S_1 for $M^2 = 0 \cdot 1$, $\alpha = 30°$.

one which corresponds to points on C_1' and C_2'. The waves, corresponding to the curve C_3', propagate all around the forcing region with asymmetric, closed wave crests. Again, this system of waves is characteristic of acoustic waves.

4. Description of Wave Pattern for Strong Compressibility

In this section, we consider strong compressibility effects (with $M^2 = 2$) on waves for $\alpha = 0$ and $30°$.

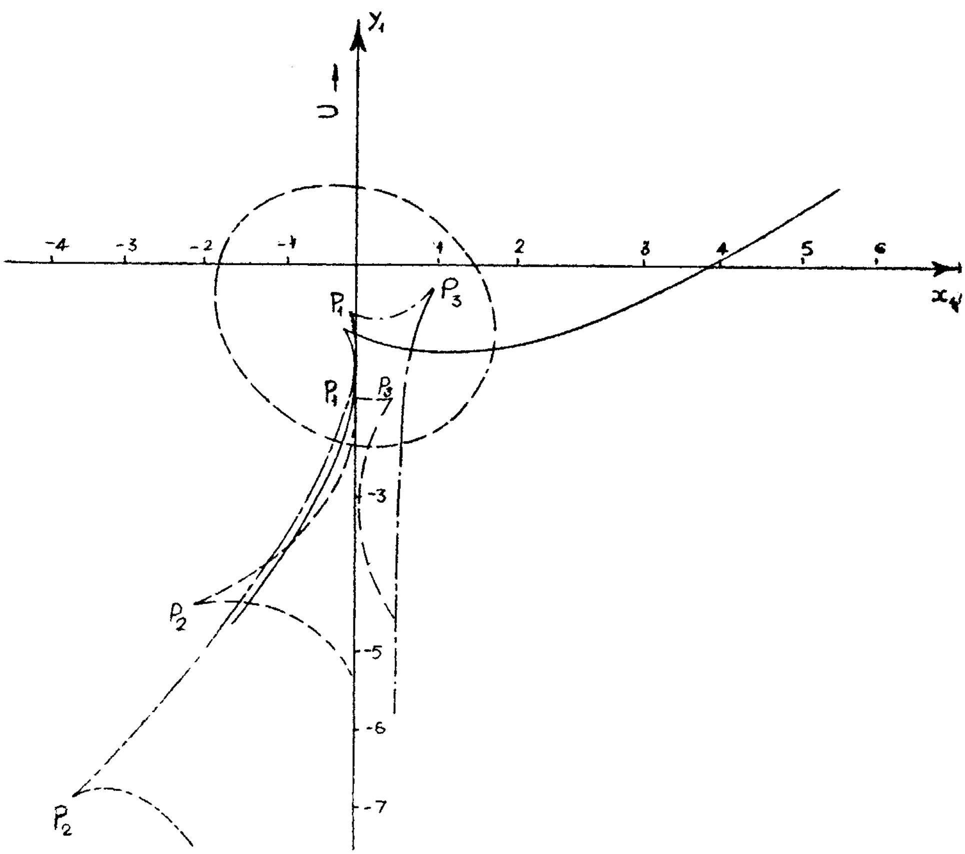

FIG. 2 (*b*). Curves of constant phase corresponding to Fig. 2 (*a*).

4.1. *Horizontally Moving Forcing Effect*

In the steady case, the waves corresponding to the curve (13) travel downstream with hyperbolic wave crests F [see Fig. 3 (*b*)]. In the regime R_1, there are four branches of the wave-number curve [see Fig. 3 (*a*)]. Corresponding to the branches C_1, C_2 the curve of constant phase consists of two parts F_1, F_2 [see Fig. 3 (*b*)]. These systems of waves represent the acoustic part of the waves propagating downstream disturbing the whole atmosphere behind the forcing region. Associated with the branches C_3 and C_4, the constant phase curves F_3 have four cusps and the waves are confined to the wedge (OR, OR').

In the regime R_2, the branch corresponding to C_2 of regime R_1 disappears. The curves corresponding C_3 and C_4 are C_3' and C_4' which are down the forcing line. The waves corresponding to the points on C_3' and C_4' propagate behind the forcing region with cusp-shaped wave crests F_2' confined to the wedge (OQ, OQ'). We see that this wedge is smaller than the wedge (OR, OR'). Hence, the fluid region disturbed in the regime R_2 is smaller than the region disturbed in the

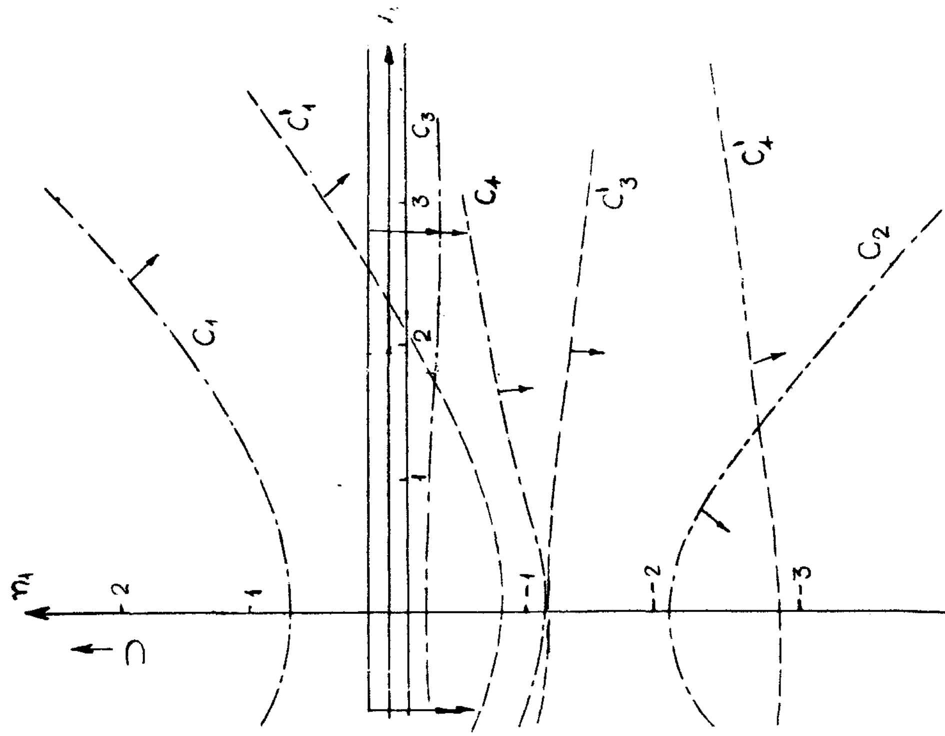

FIG. 3 (*a*). Wave-number curves S_1 for $M^2 = 2$, $\alpha = 0$.

regime R_1. The waves associated with C_1' travel downstream with U-shaped wave crests F_1' which is convex downwards. This forms the acoustic part of the system of waves.

4.2. *Forcing Effects Moving in the Direction* $\alpha = 30°$

The wave-number curves are drawn in Fig. 4 (*a*) and the curves of constant phase in Fig. 4 (*b*). In the steady case, one system of waves propagates downstream with a cusp-shaped wave crest. Another system of waves, which is characteristic of sound waves, propagates behind the forcing effect with parabola-like shape of wave crests convex upwards.

In the regime R_1, there are three systems of waves. One system consists of those waves propagating downstream with tripple-cusped wave crests [see Fig. 4 (*b*)]. We notice that there is an accumulation of wave crests to the left of the forcing line and no such accumulation to its right. There are two wave crests of parabolic shape convex upwards which are characteristic of sound waves. Thus, we notice that a single system of sound waves occurring in the steady case splits into two systems.

In the regime R_2, one system of waves propagates behind the forcing region with tripple-cusped wave crests just as in the above case. The difference is that

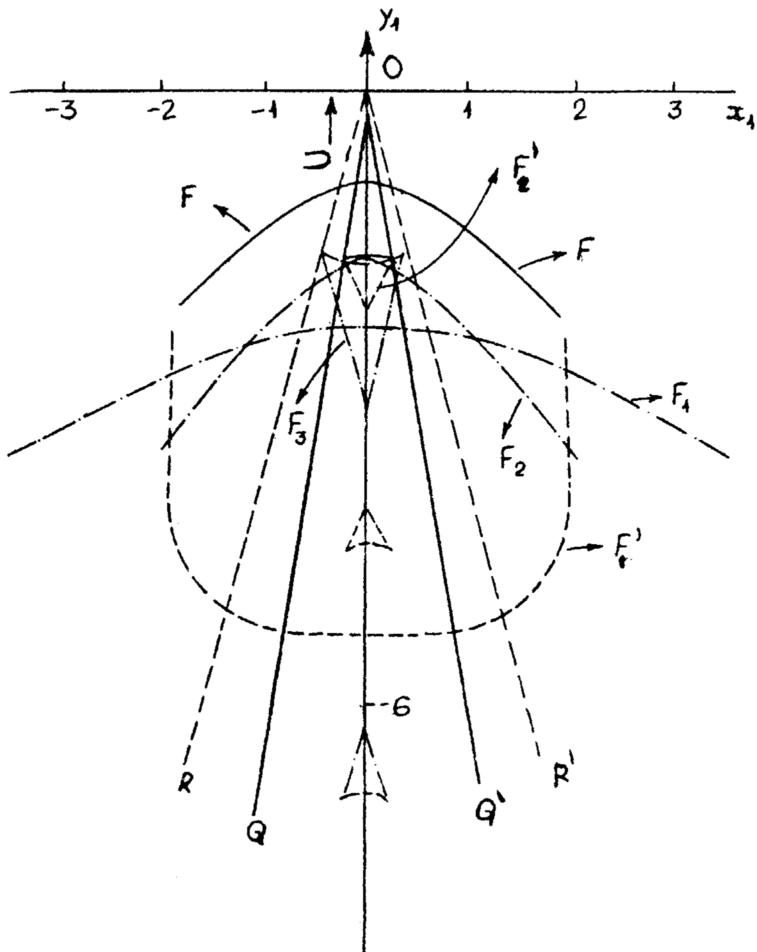

FIG. 3 (*b*). Curves of constant phase corresponding to Fig 3 (*a*).

there is a long caustic formed on the right side of the forcing line which is not present in the regime R_1. There are again two wave crests of parabola-like shape convex upwards. These crests are again characteristic of sound waves and represent two split systems.

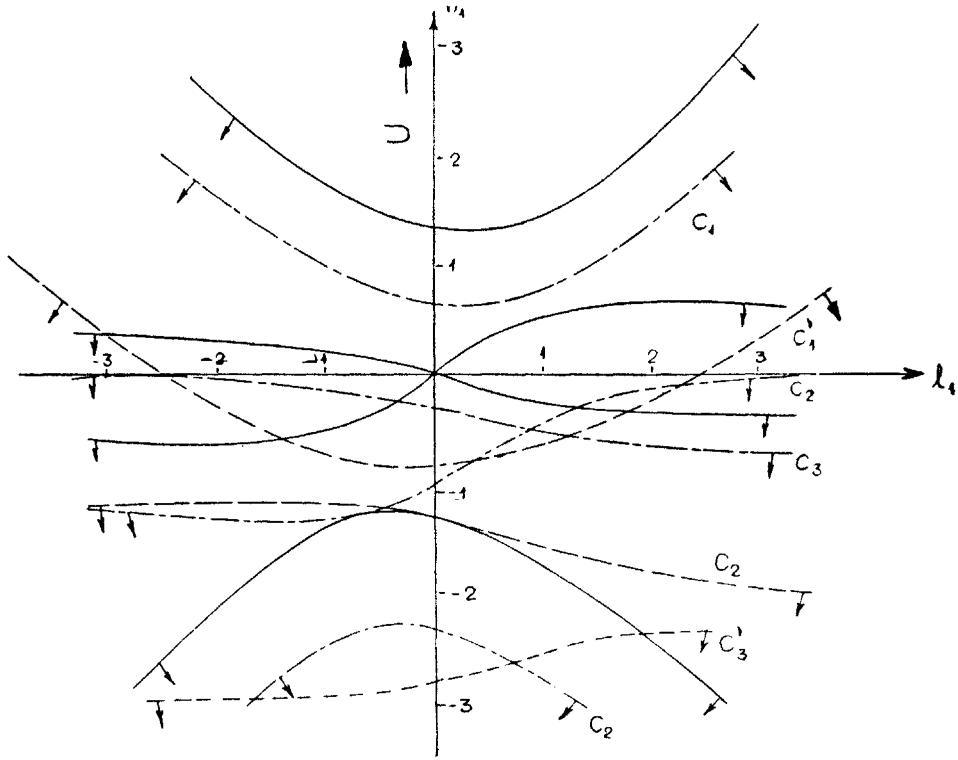

FIG. 4 (*a*). Wave-number curves for $M^2 = 2$, $\alpha = 30°$.

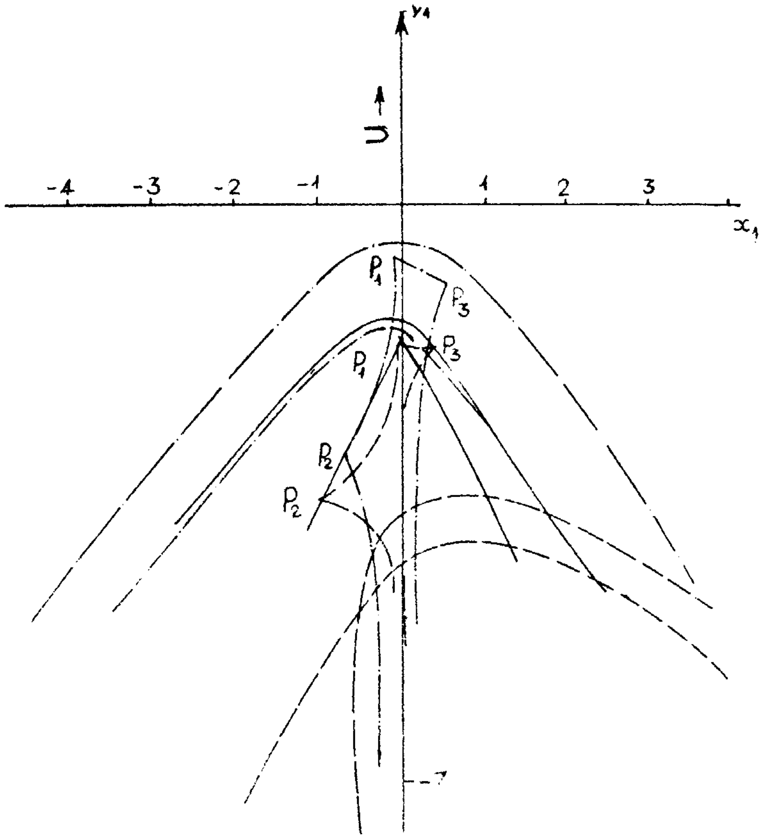

FIG. 4 (*b*). Curves of constant phase corresponding to Fig. (4 *a*).

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