ABSOLUTE TAUBERIAN CONSTANTS FOR QUASI-HAUSDORFF SERIES-TO-SERIES TRANSFORMATIONS

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This paper is concerned with introducing two inequalities of the form $\sum_{n=0}^{\infty} | b_n - a_n | \leqslant K \sum_{n=0}^{\infty} | \triangle (n a_n) |$ and $\sum_{n=0}^{\infty} | b_n - a_n | \leqslant A \sum_{n=1}^{\infty} | \triangle (1/n \sum_{\nu=1}^{n-1} \nu a_\nu)|$. where b_n denote the series-to-series quasi–Hausdorff transform, K and A are absolute Tauberian constants. The constants K and A will be determined. The Tauberian conditions used were $\sum_{n=0}^{\infty} | \triangle (n a_n) | < \infty$, $\sum_{n=1}^{\infty} | \triangle (1/n \sum_{\nu=1}^{n-1} \nu a_\nu) | < \infty$ respectively.

§ 1. *Definition* 1.1—Let $\{\mu_n\}_{n=0}^{\infty}$ be a fixed sequence of real or complex numbers. The quasi-Hausdorff transform $\{t_n\}$ of a sequence $\{s_n\}$ by means of the fixed sequence $\{\mu_{n+1}\}$ (or, in short, the (H^*, μ_{n+1}) transform) is given by

$$t_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) s_k, \qquad n = 0, 1, 2, \ldots \qquad \ldots(1.1)$$

where

$$\Delta\mu_{n+1} = \mu_{n+1} - \mu_{n+2}. \qquad \ldots(1 \cdot 2)$$

Definition 1.2—Let $\{\mu_n\}_{n=0}^{\infty}$ be a fixed sequence of real or complex numbers. Ramanujan (1953) has shown that the series-to-series quasi-Hausdorff transformation (formally equivalent) to (1.1) is given by

$$b_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_n) a_k, \qquad n = 0, 1, 2, \ldots \qquad \ldots(1.3)$$

where we write

$$t_n = b_0 + b_1 + \ldots + b_n; \qquad s_k = a_0 + a_1 + \ldots + a_k. \qquad \ldots(1.4)$$

Definition 1.3—If (1.3), (1.4) hold and if

$$\binom{k}{n}(\Delta^{k-n}\mu_n)\, s_k \to 0, \qquad \text{as} \qquad k \to \infty, \qquad \ldots(1.5)$$

then we say that the sequence $\{s_n\}$ is absolutely summable (H^*, μ_{n+1}) or summable $|H^*, \mu_{n+1}|$, if the sequence $\{t_n\}$ is of bounded variation, i.e. if

$$\sum_0^\infty |b_n| < . \infty. \qquad \ldots(1.6)$$

Anjaneyulu (1964) has shown that if

$$a_n = O\left(\frac{1}{n}\right), \qquad \ldots(1.7)$$

then under the restrictions on μ_n stated in Theorem 2.1 of this paper, the series (1.3) converges. Since (1.7) is weaker than that of the Tauberian condition

$$\sum_0^\infty |\Delta(na_n)| < \infty, \qquad \ldots(1.8)$$

then it is clear that under (1.8) the series (1.3) is convergent.

In § 2 of this paper, we introduce the following estimate :

$$\sum_0^\infty |b_n - a_n| \leqslant K \sum_0^\infty |\Delta(na_n)|, \qquad \ldots(1.9)$$

where the sequence $\{na_n\}$ is assumed to satisfy the condition (1.8), K is an absolute Tauberian constant which will be determined.

In § 3, we shall show that if the sequence $\{na_n\}$ is assumed to satisfy the Tauberian condition

$$\sum_{n=1}^\infty \left|\Delta\left(\frac{1}{n}\sum_{\nu=1}^{n-1} \nu a_\nu\right)\right| < \infty, \qquad \ldots(1.10)$$

then the series (1.3) converges, and that the following estimate holds

$$\sum_0^\infty |b_n - a_n| \leqslant A \sum_{n=1}^\infty \left|\Delta\left(\frac{1}{n}\sum_{\nu=1}^{n-1} \nu a_\nu\right)\right|, \qquad \ldots(1.11)$$

where A is an absolute Tauberian constant which will be determined. Theorems of this type have been considered in Sherif (1972), (1974), (1977). The estimate (1.9) is analogous to that obtained by Anjaneyulu (1964) for the series-to-series quasi-Hausdorff transformation involving Tauberian constants instead.

Again the estimates (1.9) and (1.11) are analogous to results shown for other summability methods by various authors. For a discussion of these analogous estimates, see Sherif (1972).

2. *Theorem* 2.1—Let $\{\mu_n\}$ be the moment sequence generated by the real function of bounded variation χ on $0 \leqslant t \leqslant 1$ so that

$$\mu_n = \int_0^1 t^n \, d\,\chi(t), \qquad \text{...(2.1)}$$

where

$$\chi(0+) = \chi(0) = 0, \qquad \chi(1) = 1, \qquad \text{...(2.2)}$$

and

$$\int_0^1 \frac{|\chi(t)|}{t} \, dt < \infty. \qquad \text{...(2.3)}$$

Let Σa_n be a series satisfying the Tauberian condition (1.8). Then (1.9) holds with

$$K = \int_0^1 \frac{|\chi(t)|}{t} \, dt. \qquad \text{...(2.4)}$$

If further, $\chi(t)$ satisfies the additional condition

$$\chi(t) \geqslant 0, \qquad \text{...(2.5)}$$

then the constant (2.4) is the best possible in the sense that (1.9) becomes false if K is replaced by any smaller constant.

For the proof of Theorem 2.1, we require the following lemma.

Lemma 2.1 (Maddox 1970, p. 167, Theorem 5, Sherif 1972, Lemma 2.1) Let

$$A_n = \sum_\nu \alpha_{n,\nu} f_\nu. \qquad \text{...(2.6)}$$

Suppose that

$$\sum_0^\infty |\alpha_{n,\nu}| \text{ is bounded.} \qquad \text{...(2.7)}$$

Let

$$K = \sup_\nu \sum_n |\alpha_{n,\nu}|. \qquad \text{...(2.8)}$$

Then

$$\sum_0^\infty |A_n| \leqslant K \sum_0^\infty |f_\nu|, \qquad \text{...(2.9)}$$

and this constant is the best possible in the sense that (2.9) becomes false if K is replaced by any smaller constant.

PROOF OF THEOREM 2.1: Since ...(2.10)

$$a_n = \frac{1}{n}\, n \,.\, a_n = -\frac{1}{n}\sum_{\nu=0}^{n-1} \Delta\,(\nu a_\nu).$$

Thus, it follows from (1.3) and (2.10) that, for $n \geqslant 1$,

$$b_n = -\sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n}\mu_n)\,\frac{1}{k}\sum_{\nu=0}^{k-1} \Delta\,(\nu a_\nu)$$

$$= -\sum_{k=n}^{\infty} \frac{1}{k}\binom{k}{n}\int_0^1 t^n(1-t)^{k-n}\,d\chi(t)\sum_{\nu=0}^{k-1} \Delta\,(\nu a_\nu). \qquad ...(2.11)$$

However, (2.10) is not valid when $n = 0$. Since $\mu_0 = 1$, we find that in this case, (2.11) can be replaced by

$$b_0 - a_0 = -\sum_{k=1}^{\infty} \frac{1}{k}\int_0^1 (1-t)^k\,d\chi(t)\sum_{\nu=0}^{k-1} \Delta\,(\nu a_\nu). \qquad ...(2.12)$$

In view of (1.8), it will follow by absolute convergence that we may invert the order of summation in (2.11) or (2.12) if we prove that

$$\sum_{k=\max(1,n)}^{\infty} \frac{1}{k}\binom{k}{n}\left|\int_0^1 t^n(1-t)^{k-n}\,d\chi(t)\right| < \infty. \qquad ...(2.13)$$

Now, integrating by parts*

$$\left\{\begin{aligned} &\frac{1}{k}\binom{k}{n}\int_0^1 t^n(1-t)^{k-n}\,d\chi(t)\\ &= \frac{1}{k}\binom{k}{n}\int_0^1 \{(k-n)\,t^n(1-t)^{k-n-1}\\ &\quad - nt^{n-1}(1-t)^{k-n}\}\,\chi(t)\,dt + \delta_{n,k} \quad \text{(say)} \end{aligned}\right. \qquad ...(2.14)$$

$$= B_{n,k} + \delta_{n,k}, \qquad \text{(say)}, \qquad ...(2.15)$$

where

$$\delta_{n,k} = \begin{cases} \dfrac{1}{n}, & (k = n \geqslant 1)\\ 0, & \text{(otherwise)}. \end{cases} \qquad ...(2.16)$$

* The extra term $\delta_{n,k}$ occurs, since in the integration by parts, the term $(1-t)^{k-n}$ does not vanish at $t = 1$ in the case $k = n$.

It, thus, follows from (2.14)-(2.16) that the left-hand side of (2.13) is equal to

$$\sum_{k=\max(1,n)}^{\infty} | B_{n,k} + \delta_{n,k} |$$

$$\leqslant \sum_{k=n+1}^{\infty} \binom{k-1}{n} \left| \int_0^1 (1-t)^{k-1-n} t^{n-1} \chi(t)\, dt \right|$$

$$+ \sum_{k=\max(1,n)}^{\infty} \binom{k-1}{n-1} \left| \int_0^1 (1-t)^{k-n} t^{n-1} \chi(t)\, dt \right| + \frac{1}{n}$$

$$\leqslant \int_0^1 | \chi(t) | t^n \left\{ \sum_{k=n+1}^{\infty} \binom{k-1}{n} (1-t)^{k-1-n} \right\} dt$$

$$+ \int_0^1 | \chi(t) | t^{n-1} \left\{ \sum_{k=n}^{\infty} \binom{k-1}{n-1} (1-t)^{k-n} \right\} dt + \frac{1}{n}$$

$$= 2 \int_0^1 \frac{| \chi(t) |}{t} dt + \frac{1}{n}, \qquad \ldots(2.17)$$

where the term $1/n$ must be omitted in the case $n = 0$. Thus, it is plain from (2.3) and (2.17) that (2.13) holds. Also, it follows from (2.11), (2.14)-(2.16) that for $n \geqslant 1$

$$b_n = - \sum_{\nu=0}^{\infty} \Delta(\nu a_\nu) \sum_{k=\max(n,\, \nu+1)}^{\infty} (B_{n,k} + \delta_{n,k}). \qquad \ldots(2.18)$$

Using (2.10) and (2.18), we have

$$b_n - a_n = \sum_{\nu=0}^{\infty} \alpha_{n,\nu} \Delta(\nu a_\nu), \qquad \text{(say)}, \qquad \ldots(2.19)$$

where

$$\alpha_{n,\nu} = \begin{cases} - \sum_{k=n}^{\infty} B_{n,k} & \text{for } (\nu \leqslant n-1) \\ - \sum_{k=\nu+1}^{\infty} B_{n,k} & \text{for } (\nu \geqslant n). \end{cases} \qquad \ldots(2.20)$$

Further, using (2.12) in place of (2.11) we see that (2.20) is still valid when $n = 0$. We now note that $B_{n,k}$ can be put in the form

$$B_{n,k} = \frac{1}{k} \binom{k}{n} \int_0^1 \{(k-n) - k(1-t)\} t^{n-1} (1-t)^{k-n-1} \chi(t)\, dt$$

$$= \binom{k-1}{n} \int_0^1 (1-t)^{k-n-1}\, t^{n-1} \chi(t)\, dt$$

$$- \binom{k}{n} \int_0^1 (1-t)^{k-n}\, t^{n-1} \chi(t)\, dt \qquad \ldots(2{\cdot}21)$$

and shall prove that

$$\binom{k}{n} \int_0^1 (1-t)^{k-n}\, t^{n-1} \chi(t)\, dt \to 0, \quad \text{as} \quad k \to \infty. \qquad \ldots(2.22)$$

Since, for fixed n,

$$\binom{k}{n} \sim \frac{1}{n!} k^n, \qquad \text{as } k \to \infty. \qquad \ldots(2.23)$$

Then (2.22) is equivalent to

$$k^n \int_0^1 (1-t)^{k-n}\, t^{n-1} \chi(t)\, dt = o(1), \quad \text{as} \quad k \to \infty. \qquad \ldots(2.24)$$

Now, for fixed n, k, the maximum of

$$k^n (1-t)^{k-n}\, t^n \qquad \ldots(2.25)$$

for $(0 \leqslant t \leqslant 1)$ occurs when $t = n/k$. It follows easily that, for fixed n eqn. (2.25) is bounded for all k and $0 \leqslant t \leqslant 1$. But for fixed n, t, (with $0 \leqslant t \leqslant 1$)

$$k^n (1-t)^{k-n} \to 0 \quad \text{as} \quad k \to \infty; \qquad \ldots(2.26)$$

hence in view of (2.3), (2.24) follows by dominated convergence. Now, using (2.20)–(2.22) it is clear that

$$\alpha_{n,\nu} = \begin{cases} 0, & \text{for } (\nu \leqslant n-1) \\ -\binom{\nu}{n} \int_0^1 (1-t)^{\nu-n}\, t^{n-1} \chi(t)\, dt, & \text{for } (\nu \geqslant n). \end{cases} \qquad \ldots(2.27)$$

Hence, the conditions of Lemma 2.1 are satisfied with

$$K = \sup_\nu S_\nu, \qquad \ldots(2.28)$$

where

$$S_\nu = \sum_{n=0}^{\nu} \binom{\nu}{n} \left| \int_0^1 (1-t)^{\nu-n}\, t^{n-1} \chi(t)\, dt \right| \qquad \ldots(2.29)$$

$$\leqslant \sum_{n=0}^{\nu} \binom{\nu}{n} \int_0^1 |\chi(t)| \, t^{n-1} (1-t)^{\nu-n} \, dt$$

$$= \int_0^1 \frac{|\chi(t)|}{t} \left\{ \sum_{n=0}^{\nu} \binom{\nu}{n} t^n (1-t)^{\nu-n} \right\} dt$$

$$= \int_0^1 \frac{|\chi(t)|}{t} \, dt. \qquad \ldots(2.30)$$

Hence, (2.4) follows from (2.28) and (2.30).

Further, if (2.5) holds, then we may omit the modulus signs in (2.29). We deduce that, in this case, there is equality in (2.30) and the final conclusion follows.

§ 3. *Theorem* 3.1—Let $\{\mu_n\}$ be the moment sequence generated by the real function of bounded variation χ on $0 \leqslant t \leqslant 1$ so that (2.1)–(2.3) hold

Let $\sum_0^\infty a_n$ be a series satisfying the Tauberian condition (1.10). Then

(i) the series (1.3) converges for each $n = 0, 1, 2, \ldots$

(ii) (1.11) holds with

$$A = \sup_\nu \left\{ |1 - \mu_\nu| + \int_0^1 (1-t)^\nu \, |d\chi(t)| + \int_0^1 \frac{|\chi(t)|}{t} \, dt \right\}. \qquad \ldots(3.1)$$

If $\chi(t)$ is non-decreasing, then (3.1) is the best possible result in the sense that (1.11) becomes false if A is replaced by any smaller constant.

Further, in this case (3.1) takes the simpler form

$$A = 2(1 - T) + \int_0^1 \frac{\chi(t)}{t} \, dt. \qquad \ldots(3.2)$$

where

$$T = \lim_{n \to \infty} \mu_n = \chi(1+) - \chi(1-). \qquad \ldots(3.3)$$

Proof: Write

$$\phi_n = -\Delta\left(\frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_\nu\right). \qquad \ldots(3.4)$$

Let

$$u_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu \; ; \quad \phi_n = u_n - u_{n-1}. \qquad \ldots(3.5)$$

Then,

$$na_n = (n+1)\, u_n - nu_{n-1}$$
$$= u_n + n\phi_k$$
$$= \sum_{\nu=1}^{n} \phi_\nu + n\phi_n. \qquad \ldots(3.6)$$

i.e. for $n \geqslant 1$,

$$a_n = \frac{1}{n} \sum_{\nu=1}^{n} \phi_\nu + \phi_n. \qquad \ldots(3.7)$$

Substituting with (3.7) in (1.3), we have

$$b_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_n) \left\{ \frac{1}{k} \sum_{\nu=1}^{k} \phi_\nu + \phi_k \right\}, \quad (n \geqslant 1). \qquad \ldots(3.8)$$

Since (3.7) is not valid when $n = 0$, we replace (3.8) in the case $n = 0$ by

$$b_0 - a_0 = \sum_{k=1}^{\infty} (\Delta^k \mu_0) \left\{ \frac{1}{k} \sum_{\nu=1}^{k} \phi_\nu + \phi_k \right\}. \qquad \ldots(3.9)$$

Let E denote the contribution to those sums of the first term in the curly brackets and let F denote the contribution of the second term.

Then

$$E = \sum_{k=\max(1,n)}^{\infty} \frac{1}{k} \binom{k}{n} \int_0^1 (1-t)^{k-n} \, d\chi(t) \, . \sum_{\nu=1}^{k} \phi_\nu. \qquad \ldots(3.10)$$

Applying the argument used in Theorem 2.1 to deduce (2.18) with $\Delta(\nu a_\nu)$ (replaced by $-\phi_{\nu+1}$), it is clear that

$$E = \sum_{\nu=1}^{\infty} \phi_\nu \sum_{k=\max(1,\nu)}^{\infty} (B_{n,k} + \delta_{n,k}). \qquad \ldots(3.11)$$

Also,

$$F = \sum_{\nu=\max(1,n)}^{\infty} \nu\phi_\nu (B_{n,\nu} + \delta_{n,\nu}), \qquad \ldots(3.12)$$

provided that this series converges. In order to establish the convergence of (3.12), it is enough to show that, for fixed n, $\nu B_{n,\nu}$ is bounded. But

$$\nu (B_{n,\nu} + \delta_{n,\nu}) = \binom{\nu}{n} \int_0^1 t^n (1-t)^{\nu-n} \, d\chi(t). \qquad \dots(3.13)$$

Since, by the argument of (2.25)

$$\nu^n (1-t)^{\nu-n} t^n$$

is, for fixed n, bounded uniformly in $0 \leqslant t \leqslant 1$, the conclusion is obvious.

(ii) Using (3.7), (3.8), (3.11) and (3.12), it follows that

$$b_n - a_n = \sum_{\nu=1}^{\infty} \psi_{n,\nu} \phi_\nu \qquad \text{(say)} \qquad \dots(3.14)$$

where

$$\psi_{n,\nu} = \begin{cases} \sum_{k=n}^{\infty} B_{n,k} & \text{for } (\nu < n) \\ \sum_{k=\nu}^{\infty} B_{n,k} + \nu B_{n,\nu} & \text{for } (\nu \geqslant n). \end{cases} \qquad \dots(3.15)$$

Now, the proof of Theorem 2.1 shows that

$$\sum_{k=\nu}^{\infty} B_{n,k} = \binom{\nu-1}{n} \int_0^1 (1-t)^{\nu-1-n} t^{n-1} \chi(t) \, dt, \qquad \dots(3.16)$$

and the argument leading to (2.21) (applied in the reverse direction) shows that

$$\nu B_{n,\nu} = \binom{\nu}{n} \int_0^1 t^n (1-t)^{\nu-n} \, d\chi(t) - \nu \delta_{n,\nu} \qquad \dots(3.17)$$

Combining (3.16) and (3.17) it follows that for $\nu \geqslant n+1$,

$$\psi_{n,\nu} = \binom{\nu}{n} \int_0^1 (1-t)^{\nu-n} t^n \, d\chi(t) + \binom{\nu-1}{n} \int_0^1 (1-t)^{\nu-1-n} t^{n-1} \chi(t) \, dt. \qquad \dots(3.18)$$

Also, for $\nu = n$,

$$\psi_{n,\nu} = \mu_n - 1. \qquad \dots(3.19)$$

Applying Lemma 2.1, we see that our Tauberian constant is now given by

$$K = \sup_\nu K_\nu \qquad \text{(say)} \qquad \dots(3.20)$$

where

$$K_{\nu} = \sum_{n=\nu}^{\infty} |\psi_{n,\nu}|. \qquad \ldots(3.21)$$

Combining (3.15), (3.18)–(3.21), it follows that

$$K_{\nu} = |1-\mu_{\nu}| + \sum_{n=0}^{\nu-1} \left| \binom{\nu}{n} \int_0^1 (1-t)^{\nu-n} t^n \, d\chi(t) + \binom{\nu-1}{n} \int_0^1 (1-t)^{\nu-1-n} t^{n-1} \chi(t) \, dt \right| \qquad \ldots(3.22)$$

$$\leqslant |1-\mu_{\nu}| + \sum_{n=0}^{\nu-1} \left\{ \binom{\nu}{n} \int_0^1 (1-t)^{\nu-n} t^n \, |d\chi(t)| + \binom{\nu-1}{n} \int_0^1 (1-t)^{\nu-1-n} t^{n-1} \, |\chi(t)| \, dt \right\}, \qquad \ldots(3.23)$$

with equality in the case in which $\chi(t)$ is non-decreasing. This completes the proof except for the simplification when $\chi(t)$ is non-decreasing.

If $\chi(t)$ is non-decreasing, we may omit the modulus signs in (3.22). We deduce that, in this case, there is equality in (3.23). We thus get

$$K_{\nu} = 1-\mu_{\nu} + \int_0^1 \left\{ \sum_{n=0}^{\nu=1} \binom{\nu}{n} (1-t)^{\nu-n} t^n \right\} d\chi(t) + \int_0^1 \left\{ \sum_{n=0}^{\nu-1} \binom{\nu-1}{n} (1-t)^{\nu-1-n} t^{n-1} \chi(t) \, dt \right.$$

$$= 1 - \mu_{\nu} + \int_0^1 (1-t^{\nu}) \, d\chi(t) + \int_0^1 \frac{\chi(t)}{t} \, dt$$

$$= 2 - 2\mu_{\nu} + \int_0^1 \frac{\chi(t)}{t} \, dt. \qquad \ldots(3.24)$$

Since $\chi(t)$ is non-decreasing, $\{\mu_n\}$ is non-increasing, and it tends to T as $n \to \infty$. It, therefore, follows from (3.24) that

$$\sup_{\nu} K_{\nu} = 2 - 2T + \int_0^1 \frac{\chi(t)}{t} \, dt. \qquad \ldots(3.25)$$

Hence, eqn. (3.2) follows from (3.20) and (3.25). This completes the proof of Theorem 3.1.

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