ON TAUBERIAN ESTIMATES FOR QUASI-HAUSDORFF TRANSFORMATIONS

by SORAYA SHERIF *and* LAYLA AWAAD, *Mathematics Department, Education College for Women, Riyadh, Saudi Arabia*

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Let t_n denote the sequence-to-sequence quasi-Hausdorff transform, [or the (H^*, μ_{n+1}) transform], such that $t_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) s_k \quad (n \geqslant 0)$, then the series-to-series quasi-Hausdorff transform b_n is defined by $b_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_n) a_k \ (n \geqslant 0)$, where $t_n = \sum_{\nu=0}^{n} b_\nu$, $s_n = \sum_{\nu=0}^{n} a_\nu$. It has been shown in this paper that the sufficient conditions under which the ordinary and the absolute Tauberian constants introduced by some authors for b_n apply also to t_n.

§ 1. Let (H^*, μ_{n+1}) be the quasi-Hausdorff transform $\{t_n\}$ of s_n defined as follows

$$t_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) s_k, \quad \text{for } (n \geqslant 0), \qquad \dots(1.1)$$

where $\Delta\mu_{n+1} = \mu_{n+1} - \mu_{n+2}$. It has been shown by Ramanujan (1953) that the series-to-series quasi-Hausdorff transform is given by

$$b_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_n) a_k \quad \text{for } (n \geqslant 0), \qquad \dots(1.2)$$

where

$$t_n = \sum_{\nu=0}^{n} b_\nu; \qquad s_k = \sum_{\nu=0}^{k} a_\nu. \qquad \dots(1.3)$$

Anjaneyulu (1964) has shown the following Tauberian estimate:

$$\limsup_{\substack{n \to \infty \\ m/n \to q}} |t_n - s_m| \leqslant A(q) \limsup_{n \to \infty} |na_n|, \qquad \dots(1.4)$$

where (1.3) holds, $(0 < q < \infty)$, $A(q)$ is a Tauberian constant, and the series $\sum_0^\infty a_n$ was assumed to satisfy the Tauberian condition

$$a_n = O\left(\frac{1}{n}\right). \qquad \text{...(1.5)}$$

Recently, Sherif (1977) (Theorems 2.1 and 3.1) has introduced two Tauberian estimates of the forms

$$\sum_0^\infty |b_n - a_n| \leqslant K \sum |\Delta(n a_n)|, \qquad \text{...(1.6)}$$

$$\sum_0^\infty \left| b_n - a_n \right| \leqslant A \sum_{n=1}^\infty \left| \Delta\left(\frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_\nu\right) \right|, \qquad \text{...(1.7)}$$

respectively, where K and A are absolute* Tauberian constants. The Tauberian conditions used were also respectively as follows:

$$\sum_0^\infty |\Delta(n a_n)| < \infty, \qquad \text{...(1.8)}$$

$$\sum_{n=1}^\infty \left| \Delta\left(\frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_\nu\right) \right| < \infty. \qquad \text{...(1.9)}$$

It is clear from the Lemma of this paper (see also Kwee 1968) that (1.1) and (1.2) are in fact equivalent [in the sense that if (1.1) converges for all n, then so does (1.2) and conversely], if and only if, for every fixed n,

$$\binom{k}{n} (\Delta^{k-n} \mu_n) s_k \to 0, \quad \text{as} \quad k \to \infty. \qquad \text{...(1.10)}$$

We remark that if the sequence $\{na_n\}$ is bounded $(C, 1)$ (and thus *a fortiori* if it is summable $|C, 1|$) then

$$s_n = O(\log n), \qquad \text{...(1.11)}$$

and hence (1.10) will hold whenever

$$\binom{k}{n} (\Delta^{k-n} \mu_n) = o(1/\log k), \qquad \text{as} \qquad k \to \infty. \qquad \text{...(1.12)}$$

In § 2 of this paper, we shall obtain sufficient conditions under which (1.12) holds; and then in § 3, we shall deduce some Tauberian results concerning $A(q)$, K and A of the estimates (1.4) (1.6) and (1.7) respectively. Similar results to those deduced in this paper have been shown by Sherif and Hanna Hamad (1976) but for the Hausdorff transformations instead.

§ 2. *Theorem*—Let $\{\mu_n\}$ be the moment sequence generated by the real function of bounded variation χ on $0 \leqslant t \leqslant 1$ so that

$$\mu_n = \int_0^1 t^n d\chi(t), \qquad \text{...(2.1)}$$

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*For the definition of (H^*, μ_{n+1}) absolute summability, see Sherif (1977).

where

$$\chi(0+) = \chi(0) = 0, \quad \chi(1) = 1. \qquad \text{...(2.2)}$$

Suppose that

$$\chi_1(t) = \int_0^t \chi(u)\,du = o\,(t/\log 1/t), \quad \text{as} \quad t \to 0. \qquad \text{...(2.3)}$$

Then (1.12) holds.

PROOF: Since, for fixed n,

$$\binom{k}{n} \sim \frac{1}{n!} k^n, \qquad \text{as} \qquad k \to \infty.$$

Then (1.12) is equivalent to

$$k^n \int_0^1 (1-t)^{k-n} t^n \, d\,\chi(t) = o\,(1/\log k) \qquad \text{...(2.4)}$$

Writing $I_n(k)$ for the expression on the left of (2.4). Integrating by parts twice we have for $k \geqslant n+2$,

$$I_{n,k} = k^n \int_0^1 (1-t)^{k-n-2} t^{n-2} F_n(k;t)\,\chi_1(t)\,dt, \qquad \text{...(2.5)}$$

where

$$F_n(k,t) = n(n-1)(1-t)^2 - 2n(k-n)\,t\,(1-t) + (k-n)(k-n-1)\,t^2.$$

Now, for fixed n and large k uniformly in $0 < t < 1/k$, we have

$$F_n(k;t) = O(1). \qquad \text{...(2.6)}$$

[In the case $n = 0$, we must use the stronger result (which holds for $n = 0$, but not in the general case)

$$F_0(k;t) = O(k^2 t^2)]. \qquad \text{...(2.7)}$$

Using (2.6) and (2.7) we see that the contribution to (2.5) of the range $(0, 1/k)$ is $o\,(1/\log k)$.

Next, for fixed n and large k uniformly in $1 > t > 1/k$,

$$F_n(k;t) = O(k^2 t^2). \qquad \text{...(2.8)}$$

If $t < 1/\sqrt{k}$, $\log(1/t) > \log\sqrt{k} = \frac{1}{2}\log k$; so the contribution to (2.5) of the range $(1/k, 1/\sqrt{k})$ is

$$o\left\{\frac{k^{n+2}}{\log k} \int_{1/k}^{1/\sqrt{k}} t^{n+1}(1-t)^{k-n-2}\,dt\right\}$$

$$= o\left\{\frac{k^{n+2}}{\log k} \int_0^1 t^{n+1}(1-t)^{k-n-2}\,dt\right\}$$

$$= o\,(1/\log k).$$

Finally for $t > 1/\sqrt{k}$,

$$\log(1-t) < -1/\sqrt{k},$$

$$\log[(1-t)^{k-n-2}] < -\sqrt{k} + O(1),$$

$$(1-t)^{k-n-2} = O(e^{-\sqrt{k}}).$$

Also, $t^n < 1$, and $\chi_1(t)$ is bounded; so the contribution to (2.5) of the range $t > 1/\sqrt{k}$ is

$$O(e^{-\sqrt{k}} k^{n+2}) = o(1/\log k).$$

Hence, the conclusion follows.

§ 3. For deducing our previously mentioned Tauberian results, we require the following lemma:

Lemma—Let any series-to-series transformation b_n be such that

$$b_n = \sum_{k=0}^{\infty} \alpha_{n,k} a_k. \qquad \text{...(3.1)}$$

Suppose that for each fixed n,

$$\alpha_{n,k} s_k \to 0, \quad \text{as} \quad k \to \infty. \qquad \text{...(3.2)}$$

Then (3.1) and the (formally equivalent) transformation

$$t_n = \sum_{k=0}^{\infty} \gamma_{n,k} s_k, \qquad \text{...(3.3)}$$

where

$$\gamma_{n,k} = \sum_{\nu=0}^{n} (\alpha_{\nu,k} - \alpha_{\nu,k+1})$$

are in fact equivalent [in the sense that if (3.1) converges for all n then so does (3.3) and conversely].

For the proof of the Lemma, see Sherif (1967).

Now, Sherif (1977) has restricted μ_n by the conditions (2.1), (2.2) and

$$\int_0^1 \frac{|\chi(t)|}{t} dt < \infty. \qquad \text{...(3.4)}$$

So, if in addition to these restrictions on μ_n, we suppose that (2.3) holds. Then under the condition (1.9), it follows by combining (1.11) with the result of the Theorem that (1.10) holds. Applying then the Lemma with

$$\alpha_{n,k} = \binom{k}{n} (\Delta^{k-n} \mu_n), \qquad \text{...(3.5)}$$

it thus follows that the absolute Tauberian constants K and A of (1.6) and (1.7) respectively, determined by Sherif (1977, Theorems 2.1 and 3.1) apply also to the sequence-to-sequence transformation (1.1).

We may also remark that since the restrictions on μ_n of Anjaneyulu (1964) were as those just mentioned by Sherif (1977), then by a similar argument to that stated above it can be shown that for every series satisfying the condition (1.5), the Tauberian constant $A(q)$ of (1.4), determined by Anjaneyulu (1964) apply also to the sequence-to-sequence transformation (1.1).

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