SOME RESULTS ON COMMON FIXED POINTS

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In this paper, some sufficient conditions for the existence of common fixed points of two mappings have been obtained.

§1. Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be contraction if

$$d(f(x), f(y)) \leqslant k\, d(x, y),\ \forall\ x, y \in X,\ 0 \leqslant k < 1. \qquad \ldots(1)$$

The well-known Banach contraction principle states that if f is a contraction on a complete metric space X, then for each $x_0 \in X$, the sequence $\{f^n(x_0)\}$ of iterates converges to a unique fixed point of f.

Kannan (1968) established an analogue of the above principle and proved the following theorem.

Theorem A—Let f be a mapping defined on a complete metric space X such that

$$d(f(x), f(y)) \leqslant \alpha\, [d(x, f(x)) + d(y, f(y))],\ \forall\ x, y \in X, 0 \leqslant \alpha < \tfrac{1}{2}. \qquad \ldots(2)$$

Then f has a unique fixed point.

In a paper Zamfirescu (1972) has also established some sufficient conditions for the existence of fixed points of a mapping f satisfying

$$d(f(x), f(y)) \leqslant \beta\, [d(x, f(y)) + d(y, f(x))],\ \forall\ x, y \in X,\ 0 \leqslant \beta < \tfrac{1}{2}. \qquad \ldots(3)$$

A mapping $f : X \to X$ is said to be contractive if

$$d(f(x), f(y)) < d(x, y),\ \forall\ x, y \in X,\ x \neq y. \qquad \ldots(4)$$

Edelstein (1962) generalized the Banach contraction principle for a more general mapping, viz. the contractive mapping, and proved the following theorem:

Theorem B—A contractive mapping f has a unique fixed point z if for some $x_0 \in X$, z is a limit point of the sequence $\{f^n(x_0)\}$ of iterates.

Later, Singh (1970) proved the following theorem for a mapping of the contractive type.

Theorem C—Let f be a mapping defined on X such that

$$d(f(x), f(y)) < \tfrac{1}{2}\, [d(x, f(x)) + d(y, f(y))],\ \forall\ x, y \in X,\ x \neq y. \qquad \ldots(5)$$

Then f has a unique fixed point z, if for some $x_0 \in X$, z is a limit point of the sequence $\{f^n(x_0)\}$ of iterates and in this case $f^n(x_0) \to z$.

The study of common fixed points for a sequence of mappings was motivated by the work of Nadler (1968) and later on it was studied by many authors. In this connection Jaggi and Sharma (1975) have proved the following generalization of Theorem B.

Theorem D—Let f_1 and f_2 be two self-mappings defined on X such that either f_1 or f_2 is continuous and

$$d(f_1(x), f_2(y)) < d(x, y), \ \forall \ x, y X, x \neq y. \qquad \ldots(6)$$

If for some $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} f_1(x_{n-1}), & \text{if } n \text{ is odd} \\ f_2(x_{n-1}), & \text{if } n \text{ is even,} \end{cases} \qquad \ldots(7)$$

has a limit point z then at least one of f_1 and f_2 possesses a fixed point in X. Further, if both f_1, f_2 have a fixed point, then z is a unique fixed point of each one of them and in this case $x_n \to z$.

Our aim in this paper is to give some more general results on common fixed points for two mappings.

§2. The following theorem is a generalization of Theorem C.

Theorem 1—Let f_1 and f_2 be two continuous self-mappings defined on X satisfying

$$d(f_1(x) f_2(y)) < \tfrac{1}{2}[d(x, f_1(x) + d(y, f_2(y))] \qquad \ldots(8)$$

for arbitrary $x, y \in X$, $x \neq y$. If for some $x_0 \in X$, the sequence $\{x_n\}$, defined in (7), has a limit point z, then at least one of f_1 and f_2 possesses a fixed point in X. Further, if both f_1 and f_2 have fixed points, then z is a unique fixed point of each of them and in this case $x_n \to z$.

PROOF: We have following two cases:

(i) $x_n = x_{n+1}$, for some n and

(ii) $x_n \neq x_{n+1}$, for all n.

Case (*i*)—Let $x_n = x_{n+1}$, for some odd n. Then $x_n = u$ is a fixed point of Further, let f_1 also have a fixed point v (say), then $v = u$ for otherwise

$$\begin{aligned} d(v, u) &= d(f_1(v), f_2(u)) \\ &< \tfrac{1}{2}[d(v, f_1(v)) + d(u, f_2(u))] \\ &= 0, \text{ a contradiction.} \end{aligned}$$

Thus, u is a common fixed point of f_1 and f_2. The uniqueness of u as a fixed point of f_1 and f_2 follows by using (8). Further, in this case $x_m = u \ \forall \ m \geq n$ and this implies $x_n \to u$ and $u = z$.

Case (ii)—Let $x_n \neq x_{n+1}$ for all n. For odd n, we have

$$d(x_n, x_{n+1}) = d(f_1(x_{n-1}), f_2(x_n))$$
$$< \tfrac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$\Rightarrow \quad d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Similarly, $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for even n. Thus, $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing, bounded sequence and so there exists a real number r such that

$$d(x_n, x_{n+1}) \to r.$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging to z. Choose a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that n_{k_i}'s are either all even or all odd. Let all n_{k_i}'s be even. Since $x_{n_{k_i}} \to z$, the continuity of f_1, f_2 implies that

$$f_1(x_{n_{k_i}}) = x_{n_{k_i}+1} \to f_1(z)$$

and

$$f_2(x_{n_{k_i}+1}) = x_{n_{k_i}+2} \to f_2(f_1(z)).$$

If $z \neq f_1(z)$, then

$$d(z, f_1(z)) = \lim_{i\to\infty} d(x_{n_{k_i}}, x_{n_{k_i}+1})$$
$$= r$$
$$= \lim_{i\to\infty} d(x_{n_{k_i}+1}, x_{n_{k_i}+2})$$
$$= d(f_1(z), f_2(f_1(z)))$$
$$< \tfrac{1}{2}[d(z, f_1(z)) + d(f_1(z), f_2(f_1(z)))].$$

Thus, $d(z, f_1(z)) < d(z, f_1(z))$, a contradiction.

Therefore, $f_1(z) = z$. Similarly, by choosing n_{k_i}'s all odd we see that $f_2(z) = z$. Consequently z is a fixed point of either f_1 or f_2. As in case (i) we can further show that z is a unique fixed point of both f_1 and f_2.

Further, we see that

$$d(x_n, z) < d(x_{n-1}, z) \text{ for all } n.$$

So $\{d(x_n, z)\}$ is a monotonically decreasing bounded sequence and, thus is convergent. Also z is a limit point of $\{x_n\}$. Hence $x_n \to z$.

Now, we give an example in which the condition (8) of Theorem 1 is satisfied whereas the condition (6) of Theorem D is not satisfied.

Example—Let $X = [0, 1]$ and let f_1 and f_2 be two self-mappings defined on X as follows:

$$f_1(x) = \frac{x}{4} \text{ and } f_2(x) = \frac{x}{5}, \quad x \in [0, 1].$$

It is easy to verify that the conditions of Theorem 1 are satisfied with 0 as the unique fixed point of f_1 and f_2. That the condition (6) is not satisfied can be seen by taking $x = 1$ and $y = 31/32$.

Now we state a theorem whose proof runs on the lines similar to that of Theorem 1.

Theorem 2—Let f_1 and f_2 be two continuous self-mappings defined on X such that

$$d(f_1(x), f_2(y)) < \tfrac{1}{2}[d(x, f_2(y)) + d(y, f_1(x))] \qquad \ldots(9)$$

for arbitrary $x, y \in X$, $x \neq y$. If a sequence $\{x_n\}$ as defined in (7) has a limit point z, then the assertion of Theorem 1 holds.

The next theorem is again a generalization of Theorem C. Only a brief outline of the proof is given.

Theorem 3—Let f_1 and f_2 be two self-mappings defined on X such that $f_1 f_2$ and $f_2 f_1$ are continuous and

$$d(f_1 f_2(x), f_2 f_1(y)) < \tfrac{1}{2}[d(x, f_1 f_2(x)) + d(y, f_2 f_1(y))], \qquad \ldots(10)$$

for arbitrary $x, y \in X$, $x \neq y$. If for some $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} f_1 f_2(x_{n-1}), & \text{if } n \text{ is odd,} \\ f_2 f_1(x_{n-1}), & \text{if } n \text{ is even,} \end{cases} \qquad \ldots(11)$$

has a limit point z, then z is a unique common fixed point of f_1 and f_2 such that $x_n \to z$.

PROOF: It follows by Theorem 1 that at least one of $f_1 f_2$ and $f_2 f_1$ has a fixed point u(say). Without loss of generality we assume that u is a fixed point of $f_1 f_2$, i.e. $f_1 f_2(u) = u$. Then

$$f_2(f_1 f_2(u)) = f_2(u),$$

which implies that $f_2(u)$ is a fixed point of $f_2 f_1$.

Now $f_2(u) = u$, otherwise by (10), we have

$$\begin{aligned} d(u, f_2(u)) &= d(f_1 f_2(u), f_2 f_1(f_2(u))) \\ &< \tfrac{1}{2}[d(u, f_1 f_2(u)) + d(f_2(u), f_2 f_1(f_2(u)))] \\ &= 0, \text{ a contradiction.} \end{aligned}$$

Hence, $f_2(u) = u$.

Again, $f_1(u) = f_1(f_2(u)) = u$.

Therefore, u is a common fixed point of f_1 and f_2. The uniqueness of u as a common fixed point of f_1 and f_2 follows easily by using (10).

The last assertion that $x_n \to z$ follows by using the technique as applied in Theorem 1 and then $u = z$. This completes the proof of the theorem.

We now state a theorem which follows by using Theorem 2 and Theorem 3.

Theorem 4—Let f_1 and f_2 be two self-mappings defined on X such that $f_1 f_2$ and $f_2 f_1$ are continuous and

$$d(f_1 f_2(x), f_2 f_1(y)) < \tfrac{1}{2}[d(x, f_2 f_1(y)) + d(y, f_1 f_2(x))], \qquad \ldots(12)$$

for arbitrary $x, y \in X$, $x \neq y$. If the sequence $\{x_n\}$ as defined in (11) has a limit point z, then z is a unique common fixed point of f_1 and f_2 and in this case $x_n \to z$.

We remark that Theorem 1 and Theorem 3 reduce to Theorem C if we take $f_1 = f_2 = f$ and $f_2 = I$, $f_1 = f$ respectively and if we take these conditions in Theorem 2 and Theorem 4 we get the corresponding results for a mapping f satisfying

$$d(f(x), f(y)) < \tfrac{1}{2}[d(x, f(y)) + d(y, f(x))], \quad x, \; y \in X, \; x \neq y. \qquad \ldots(13)$$

Finally we state a theorem which generalizes Theorem 1 and Theorem 2 of this paper.

Theorem 5—Let f_1 and f_2 be two continuous self-mappings defined on X satisfying either of the following conditions:

$$d(f_1(x), f_2(y)) < \alpha d(x, f_1(x)) + (1-\alpha)\, d(y, f_2(y)), \; 0 < \alpha < \tfrac{1}{2}; \qquad \ldots(14)$$

$$d(f_1(x), f_2(y)) < \alpha d(x, f_2(y)) + (1-\alpha)\, d(y, f_1(x)), \; 0 < \alpha < \tfrac{1}{2}; \qquad \ldots(15)$$

$$d(f_1(x), f_2(y)) < \alpha d(x, f_1(x)) + \beta d(y, f_2(y)) + \gamma d(x, y),$$
$$\alpha > 0, \beta > 0, \gamma \geqslant 0, \alpha + \beta + \gamma = 1, 2\alpha + \gamma < 1; \qquad \ldots(16)$$

$$d(f_1(x), f_2(y)) < \alpha\, d(x, f_2(y)) + \beta d(y, f_1(x)) + \gamma d(x, y),$$
$$\alpha > 0, \; \beta > 0, \; \gamma \geqslant 0, \; \alpha + \beta + \gamma = 1, \; 2\alpha + \gamma < 1; \qquad \ldots(17)$$

for arbitrary $x, y \in X$, $x \neq y$. Further, let the sequence $\{x_n\}$ as defined in (7) has a limit point z. Then the assertion of Theorem 1 holds.

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