RADIUS OF STARLIKENESS OF CLOSE-TO-SPIRALLIKE FUNCTIONS

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(*Received* 9 *June* 1976)

In this paper we obtain the radius of univalence and starlikeness of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular in $|z| < 1$ and satisfy the condition $\text{Re}\left[\frac{f(z)}{g(z)}\right] > \beta \, (0 < \beta < 1)$, $|z| < 1$, where $g(z)$ is α-spiral in $|z| < 1$.

§ 1. If $g(z) = z + b_2 z^2 + \ldots$ is regular in the unit disc $E = [z; |z| < 1]$ and satisfies the condition

$$\text{Re}\left[e^{i\alpha} \frac{zg'(z)}{g(z)}\right] > 0, \; z \in E, \qquad \ldots(1)$$

for some real α, $|\alpha| < \pi/2$ then $g(z)$ is said to be spirallike (Libera 1967).

Such functions are known to be univalent in E (Spacek 1933). Let $A(\alpha)$ denote the class of functions $g(z)$ satisfying the above conditions for a given α.

A function $f(z) = z + a_2 z^2 + \ldots$ regular in E is said to be close-to-star (Reade 1955) if there exists a regular univalent starlike function $h(z)$ in E such that

$$\text{Re}\left[\frac{f(z)}{h(z)}\right] > 0, \; z \in E. \qquad \ldots(2)$$

The purpose of this paper is to generalize the concept of close-to-starlikeness by replacing $h(z)$ in (2) with a spirallike function and to obtain the radius of starlikeness of this class of functions.

Definition—A function $f(z) = z + \sum_{n=n}^{\infty} a_2 z^n$ regular in E is close-to-spiral like if there exists a function $g(z)$ in $A(\alpha)$ such that

$$\text{Re}\left[\frac{f(z)}{g(z)}\right] > 0, \; z \in E. \qquad \ldots(3)$$

On taking $\alpha = 0$, we see that every close-to-spiral function reduces to a close-to-star function. It was shown by Reade (1955) that close-to-star functions are not always univalent in E; consequently it follows that close-to-spiral functions need not be univalent in E.

We introduce a more general class of functions. Let $S(\alpha, \beta)$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

regular in E and satisfying the condition

$$\operatorname{Re}\left[\frac{f(z)}{g(z)}\right] > \beta,\ 0 \leqslant \beta < 1,\ z \in E, \qquad \ldots(4)$$

where $g(z) \in A(\alpha)$.

We call functions in $S(\alpha, \beta)$ close-to-spirallike of order β. $S(\alpha, 0)$ coincides with the class of close-to-spiral functions.

Let P denote the class of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

regular in E and satisfying the condition $\operatorname{Re} p(z) > 0$, $z \in E$. Thus we have $f(z) \in S(\alpha, \beta)$ if and only if there exists a $g(z) \in A(\alpha)$ and $p(z) \in P$ such that

$$\frac{f(z)}{g(z)} = (1 - \beta) p(z) + \beta. \qquad \ldots(5)$$

Differentiating (5) logarithmically we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{[\beta/(1-\beta)] + p(z)} \quad . \qquad \ldots(6)$$

§ 2. In order to determine the radius of starlikeness of the class $S(\alpha, \beta)$ we need the following lemmas.

Lemma 1—If $g(z) \in A(\alpha)$ and $|z| = r$, then

$$\operatorname{Re}\left[\frac{zg'(z)}{g(z)}\right] \geqslant \frac{1 - 2r\cos\alpha + r^2\cos 2\alpha}{1 - r^2} \quad . \qquad \ldots(7)$$

The bound is sharp.

PROOF: Since $g(z) \in A(\alpha)$, $\operatorname{Re}\left[e^{i\alpha}\frac{zg'(z)}{g(z)}\right] > 0, z \in E$ and hence there exists a $p(z) \in P$ such that

$$\frac{zg'(z)}{g(z)} = e^{-i\alpha}[\cos\alpha \,.\, p(z) + i\sin\alpha] \qquad \ldots(8)$$

Using the fact that $p(z)$ is sub-ordinate to $\frac{1+z}{1-z}$, we obtain (Nehari 1954, p. 173).

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{2r}{1-r^2},\ |z| = r. \qquad \ldots(9)$$

Equations (8) and (9) yield

$$\left|\frac{zg'(z)}{g(z)} - \frac{1+e^{-2i\alpha}r^2}{1-r^2}\right| \leqslant \frac{2r\cos\alpha}{1-r^2},\ z \in E \qquad \ldots(10)$$

which gives (7).

The equality sign in (7) is attained at the point $z = r$ for the function

$$g(z) = \frac{z}{(1-e^{i\lambda} \cdot z)^{2\cos\alpha\, e^{-i\alpha}}}$$

where

$$\tan\lambda = -\frac{(1-r^2)\sin\alpha}{2r-(1+r^2)\cos\alpha}. \qquad \ldots(11)$$

Lemma 2—If $p(z) \in P$ and $|z| = r$, then

$$\mathrm{Re}\left[\frac{zp'(z)}{p(z)+\beta/(1-\beta)}\right]$$

$$\geq \begin{cases} \dfrac{-2(1-\beta)r}{(1+r)\left(1+(2\beta-1)r\right)} & \text{for } R_0 \leqslant R_1, \\ \dfrac{1}{1-\beta}\left[2\left(\dfrac{\beta\left(1+(1-2\beta)r^2\right)}{1-r^2}\right)^{\frac{1}{2}} - \dfrac{1+\beta+(1-3\beta)r^2}{1-r^2}\right] & \text{for } R_0 \geqslant R_1, \end{cases} \qquad \ldots(12)$$

where

$$R_0^2 = \frac{\beta}{(1-\beta)^2} \cdot \frac{1+(1-2\beta)r^2}{1-r^2}, \quad R_1 = \frac{1+(2\beta-1)r}{(1-\beta)(1+r)}.$$

PROOF: Since $p(z) \in P$, we can write it in the form

$$p(z) = \frac{1-\omega(z)}{1+\omega(z)} \qquad \ldots(13)$$

where $w(z)$ is regular in E with $\omega(o) = 0$ and $|\omega(z)| < 1$.

The function

$$\phi(z) = \frac{\omega(z)}{z} \qquad \ldots(14)$$

is also bounded and analytic in E and for such functions we have (Nehari 1954, p. 168).

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2}. \qquad \ldots(15)$$

From (13), (14) and (15) we obtain

$$|zp'(z) - \tfrac{1}{2}(p^2(z)-1) \leqslant \frac{r^2|p(z)+1|^2 - |p(z)-1|^2}{2(1-r^2)},$$

which yields

$$\mathrm{Re}\left[\frac{zp'(z)}{p(z)+\beta/(1-\beta)}\right] \geqslant \frac{1}{2}\,\mathrm{Re}\,\frac{p^2(z)-1}{p(z)+\beta/(1-\beta)}$$

$$-\frac{r^2|p(z)+1|^2 - |p(z)-1|^2}{2(1-r^2)\left|p(z)+\dfrac{\beta}{1-\beta}\right|}.$$

Put $p(z) + \frac{\beta}{1-\beta} = \mathrm{Re}^{i\theta}$ and let $S(R, \theta)$ denote the resulting expression.

Then

$$S(R,\theta) = -\frac{\beta}{1-\beta} + \frac{(1-(1-2\beta)^2 r^2)}{2(1-\beta)^2(1-r^2)R} + \frac{R}{2}$$
$$+ \cos\theta\left[\frac{R}{2} + \frac{2\beta-1}{2(1-\beta)^2}\cdot\frac{1}{R} - \frac{1+(1-2\beta)r^2}{(1-\beta)(1-r^2)}\right]. \quad \ldots(16)$$

Differentiating (16) with respect to θ, keeping R fixed, we get

$$\frac{\partial s}{\partial \theta} = \sin\theta \,.\, T(R),$$

where

$$T(R) = \frac{1+(1-2\beta)r^2}{(1-\beta)(1-r^2)} + \frac{1-2\beta}{2(1-\beta)^2}\cdot\frac{1}{R} - \frac{R}{2}$$

We have to determine the minimum of $S(R, \theta)$ inside the circle

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{2r}{1-r^2}\,.$$

It is easy to see that

$$R_1 \leqslant R \leqslant R_2,$$

where

$$R_2 = \frac{1+(1-2\beta)r}{(1-\beta)(1-r)}\,.$$

An easy calculation shows that $T'(R)$ is negative for $R_1 \leqslant R \leqslant R_2$ and $0 \leqslant \beta < 1$. Hence $T(R)$ is a monotone decreasing function of R. Therefore, Min $T(R)$ is attained at $R = R_2$ and equals

$$\frac{1-(1-2\beta)r^2}{(1+r)(1+(1-2\beta)r)} > 0.$$

Hence the minimum of $S(R, \theta)$ inside the circle

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{2r}{1-r^2}$$

is attained at $\theta = 0$. On putting $\theta = 0$ in (16) we get

$$L(R) \equiv S(R, 0) = \frac{-\beta}{1-\beta} - \frac{1+(1-2\beta)r^2}{(1-\beta)(1-r^2)} + R$$
$$+ \frac{\beta}{(1-\beta)^2}\cdot\frac{1+(1-2\beta)r^2}{(1-r^2)}\cdot\frac{1}{R},$$

where $R_1 < R < R_2$.

The absolute minimum of $L(R)$ in $(0, \infty)$ is attained at $R = R_0$ and equals

$$L(R_0) = \frac{1}{1-\beta}\left[2\left(\frac{\beta\left(1+(1-2\beta)r^2\right)}{1-r^2}\right)^{\frac{1}{2}} - \frac{1+\beta+(1-3\beta)r^2}{1-r^2}\right]. \qquad ...(17)$$

It is easy to see that R_0 is always less than R_2 but R_0 is not always greater than R_1. When $R_0 \notin [R_1, R_2]$, the minimum of $S(R, \theta)$ on the segment $[R_1, R_2]$ is attained at $R = R_1$ and equals

$$L(R_1) = \frac{-2(1-\beta)r}{(1+r)\left(1+(2\beta-1)r\right)}. \qquad ...(18)$$

$L(R_0) = L(R_1)$ for such values of β for which $R_0 = R_1$. Equations (17) and (18) yield (12).

The equality signs in (12) are attained respectively for the first and second inequality for the functions given by the following equations:

$$p(z) = \frac{1-z}{1+z}, \qquad ...(19)$$

$$p(z) = \frac{1-z^2}{1-2z\cos\theta+z^2}, \qquad ...(20)$$

where $\cos\theta$ is determined from

$$\frac{1-2\beta\cos\theta\,.\,r+(2\beta-1)r^2}{1-2\cos\theta\,.\,r+r^2} = R_0. \qquad ...(21)$$

Theorem—Let $f(z) \in S(\alpha, \beta)$. Let β_0 denote the smallest positive root of the equation

$$A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 = 0, \qquad ...(22)$$

where

$$A_4 = 8(9\cos^2\alpha - 3\cos\alpha - 5)(3\cos^2\alpha - \cos\alpha - 2),$$

$$A_3 = 4(9\cos^6\alpha - 18\cos^5\alpha - 100\cos^4\alpha + 88\cos^3\alpha + 115\cos^2\alpha - 51\cos\alpha - 42),$$

$$A_2 = 2(3\cos^8\alpha - 10\cos^7\alpha - 23\cos^6\alpha + 70\cos^5\alpha + 106\cos^4\alpha - 154\cos^3\alpha - 134\cos^2\alpha + 83\cos\alpha + 58),$$

$$A_1 = -8\cos^8\alpha + 27\cos^7\alpha + 12\cos^6\alpha - 91\cos^5\alpha - 26\cos^4\alpha + 127\cos^3\alpha + 50\cos^2\alpha - 62\cos\alpha - 32,$$

$$A_0 = (2\cos^4\alpha - 3\cos^3\alpha - 3\cos^2\alpha + 3\cos\alpha + 2)(\cos^4\alpha - 2\cos^3\alpha - \cos^2\alpha + 2\cos\alpha + 1).$$

Then (i) for $0 \leqslant \beta \leqslant \beta_0$, the radius of starlikeness of $f(z)$ is given by the smallest positive root of the equation

$$(2\beta - 1) \cos 2\alpha \,.\, r^3 + [\cos 2\alpha + 2(1 - 2\beta) \cos \alpha + 2(1 - \beta)] r^2 + (4\beta - 3 - 2\cos\alpha) r + 1 = 0; \quad \ldots(23)$$

(ii) for $\beta_0 \leqslant \beta < 1$, the radius of starlikeness of $f(z)$ is given by the smallest positive root of the equation

$$[(1 - \beta)\cos^4 \alpha - (1 - 2\beta)\cos 2\alpha] r^4 - [(1 - \beta)\cos 2\alpha + 3\beta - 1] \cos\alpha \,.\, r^3 + [(1 - \beta)\cos^2\alpha + 2\beta \sin^2\alpha] r^2 + 2\beta\cos\alpha \,.\, r - \beta = 0. \quad \ldots(24)$$

The bounds are sharp.

PROOF: By eqns. (6), (7) and (12) we obtain

$$\operatorname{Re}\left[\frac{z f'(z)}{f(z)}\right] \geqslant \begin{cases} \dfrac{r^2\cos 2\alpha - 2r\cos\alpha + 1}{1 - r^2} - \dfrac{2(1-\beta) r}{(1 + r)\left(1 + (2\beta - 1) r\right)} & \text{for } R_0 \leqslant R_1, \\ \dfrac{r^2\cos 2\alpha - 2r\cos\alpha + 1}{1 - r^2} + \dfrac{1}{1 - \beta}\left\{2\left(\dfrac{\beta\left(1 + (1 - 2\beta) r^2\right)}{1 - r^2}\right)^{\frac{1}{2}} \right. \\ \left. \quad - \dfrac{(1 + \beta) + (1 - 3\beta) r^2}{1 - r^2}\right. & \text{for } R_0 \geqslant R_1. \end{cases} \quad \ldots(25)$$

Hence the radii of starlikeness of $f(z)$ are determined from the equations,

$$\frac{1 - 2r\cos\alpha + r^2\cos^2\alpha}{1 - r^2} - \frac{2(1 - \beta) r}{(1 + r)\left(1 + (2\beta - 1) r\right)} = 0, \quad \ldots(26)$$

$$\frac{1 - 2r\cos\alpha + r^2\cos^2\alpha}{1 - r^2} + \frac{1}{1 - \beta} \cdot \left[2\left(\frac{\beta\left(1 + (1 - 2\beta) r^2\right)}{1 - r^2}\right)^{\frac{1}{2}} - \frac{1 + \beta + (1 - 3\beta) r^2}{1 - r^2}\right] = 0, \quad \ldots(27)$$

Equations (26) and (27) reduce to eqns. (23) and (24) respectively. Also the two minima given by (25) become equal to each other for such β $(0 \leqslant \beta < 1)$ for which

$$R_0 = R_1. \quad \ldots(28)$$

Since we are interested in those real roots of (23) and (24) for which $0 < r < 1$, we see that the radii of starlikeness of $f(z)$ are given by the smallest positive roots of (23) and (24).

For some value of β, the radii of starlikeness of $f(z)$ given by (23) and (24) may become equal. Such values of β will be obtained on eliminating r from (23) and (28) and are the roots of the equation

$$F(\beta) = A_4\beta^4 + A_3\beta^3 + A_2\beta^2 + A_1\beta + A_0 = 0. \quad \ldots(29)$$

Notice that $F(0) = A_0 [(2\cos^4\alpha - 3\cos^2\alpha + 2) + 3\cos\alpha\sin^2\alpha](\sin^2\alpha + \cos^4\alpha + 2\cos\alpha\sin^2\alpha) > 0$ for $|\alpha| < \pi/2$.

$$F(1) = A_4 + A_3 + A_2 + A_1 + A_0$$
$$= (\cos\alpha - 1)[2 + 7\cos\alpha\sin^2\alpha + 4\cos^2\alpha\sin^2\alpha + 3\cos^5\alpha] < 0$$

for $|\alpha| < \pi/2$.

Hence there exists a smallest positive root $\beta_0 (0 < \beta_0 < 1)$ of the equation $F(\beta) = 0$.

Functions given by (11), (19) and (20) show that the bounds are sharp.

Corollary 1—Each function $f(z)$ in $S(\alpha, 0)$ maps

$$|z| < \frac{1 + \cos\alpha - \sqrt{1 + \sin^2\alpha + 2\cos 2\alpha}}{\cos 2\alpha}$$

onto a starlike domain.

The result follows immediately on taking $\beta = 0$ in (23).

Corollary 2—Every close-to-star function is starlike in $|z| < 2 - \sqrt{3}$.

This is a result due to MacGregor (1963) and follows from Corollary 1 on taking $\alpha = 0$.

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