INDUCED POTENTIAL PROBLEM IN THREE DIMENSIONS

by B. M. NAYAR, *Department of Applied Mathematics, Thapar College of Engineering, Patiala*

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Properties of the kernel of Fredholm-Poincarè integral equation that arises in an induced potential problem in three dimensions are discussed. In classical potential theory this kernel is noted to be discontinuous, unbounded and at points, infinite. Following a functional analytic approach it is shown that the kernel is continuous and bounded in a certain normed space.

Results analogous to Hilbert's quadratic and bilinear expansions are given in terms of the eigen set of the kernel. Also its degenerate form of finite rank is shown to be bounded in norm, the bounds being given in terms of its eigen-values. Furthermore, convergence or order of growth of certain infinite series is noted. The solution of the integral equation as a Mittag-Leffler series in terms of Poincarè's fundamental functions concludes the paper.

1. INTRODUCTION

The Robin-Poincarè problem, the induced potential problem or more commonly the Neumann boundary value problem in three dimensions leads to the solution of the following Fredholm-Poincaré integral equation

$$\phi = f + \lambda K\phi, \qquad ...(1.1)$$

where f is the value of the normal derivative of the potential on the closed smooth surface S of class B (Kellogg 1929, Nayar 1967) with respect to which the problem is formulated and λ is the parameter.

The solution ϕ of (1.1) corresponds to the density of the single layer whose potential gives rise to the harmonic function sought. It can also be expressed in terms of the solution of the corresponding homogeneous equation

$$\phi = \lambda K\phi. \qquad ...(1.2)$$

If r_{pq} is the distance between the points p and q of S and n_p is the outward normal at p, then the kernel K of (1.1) is

$$K(p,q) = \frac{\partial}{\partial n_p} G(p,q) \qquad ...(1.3)$$

where

$$G(p,q) = \frac{1}{2\pi r_{pq}} . \qquad ...(1.4)$$

The theory of integral equations with continuous and bounded kernels is fairly developed (Fredholm 1903). In classical potential theory, in three dimensions, K is noted to be discontinuous, unbounded and becomes infinite in the ordinary sense when $p \to q$. Kellogg (1929) has examined the discontinuity relevant to K and has shown that the result for continuous kernels apply, since its third iterate K^3 is bounded and continuous function of p and q.

The above kernel also belongs to the extensive class of symmetrizable kernels (Blumenfeld and Mayer 1914) and is symmetrizable on the left by a positive definite and symmetric transformation G given by (1.4). In fact K is fully symmetrizable by it (Nayar 1967).

Howland and Vaillancourt (1966) have shown that the second iterate K^2 is completely continuous on the pre-Hilbert space of continuous functions on the bounding surface in the ordinary sense with L^2 norm. Mikhlin (1960) also has established that a weakly singular, non-symmetric transformation is completely continuous with respect to a similar norm. In an alternate approach (Nayar 1967) depending on maximum principle, expansion theorems and arguments due to Riesz and Nagy (1955) it has been shown that K is completely continuous with respect to the Dirichlet norm. These arguments can be extended to show that K is continuous and bounded with respect to that norm as well.

2. Dirichlet Norm and the Space H

Let H be a class of real functions $f(p)$ defined and continuous for $p \in S$, then

$$K : H \to H \text{ and } G : H \to H$$

define linear transformations in H.

If $f,\ g \in H$, then in usual notation,

$$(f, g) = \int_s f(p)\, g\,(p)\, dS_p.$$

We define a scalar product in H as

$$[f, g] = (Gf, g). \qquad \ldots(2.1)$$

G is symmetric and positive definite, i.e.

$$(Gf, g) = (f, Gg); (Gf, f) > 0.$$

GK is a symmetric transformation and K is symmetric in the new scalar product, i.e.

$$[Kf, g] = [f, Kg] \qquad \ldots(2.2)$$

The Dirichlet norm of $f \in H$ is defined as

$$\| f \| = [f, f]^{1/2} = (J + J')^{1/2} > 0, \qquad \ldots(2.3)$$

where J and J' are Dirichlet integrals for regions R and R', the interior and exterior to S respectively and are essentially positive quantities. Furthermore,

$$[Kf, f] = J - J', \quad \text{...(2.4)}$$

thence

$$[Kf, f] < [f, f]. \quad \text{...(2.5)}$$

In view of the fact that K satisfies the Fredholm alternative for all values of λ and that it has a discrete spectrum with no finite point of accumulation, the maximum principle helps in establishing the existence of eigen-values and eigen-functions of K (Nayar 1967).

The least upper bound of

$$M(f) = \frac{[Kf, f]}{[f, f]} = \frac{J - J'}{J + J'} < 1 \; \{f : [f, f] = 1; \; f \in H\} \quad \text{...(2.6)}$$

is a positive eigen-value of K where the extremizing sequence $\{f\}$ is normalized with respect to the scalar product. Thus there exists a sequence $\{f_n\}$ such that

$$[Kf_n, f_n] \to \lambda_1^{-1}, \; [f_n, f_n] = 1. \quad \text{...(2.7)}$$

λ_1^{-1} is an eigen-value of K.

If ϕ_1 denotes the function for which the least upper bound is attained, then ϕ_1 defines the eigen-function corresponding to the eigen-value λ_1^{-1} and

$$\phi_1 = \lambda_1 K\phi_1.$$

Since K is fully symmetrizable i.e., $GK\phi = 0$ if and only if $\phi \equiv 0$, the following more general spectral representation results:

The set $\{\phi_i, \lambda_i^{-1}\}$ of eigen-functions and eigen-values of K is denumerably infinite where $\lambda_i^{-1} \to 0$ and $\{\phi_i\}$ is the maximal set of linearly independent orthogonal set of eigen-functions ϕ_i corresponding to positive eigen-values

$$\lambda_1^{-1} > \lambda_2^{-1} > \cdots > \lambda_n^{-1}. \quad \text{...(2.8)}$$

3. Expansion Results : Complete Continuity of K

Let

$$f_i = [f, \phi_i], \; i = 1, 2, \ldots \quad \text{...(3.1)}$$

denote the generalized Fourier coefficients of $f \in H$ with respect to the spectrum $\{\phi_i, \lambda_i^{-1}\}$, then the following series are absolutely and uniformly convergent:

$$\left.\begin{aligned} &\text{(i) } f = \Sigma f_i \phi_i \\ &\text{(ii) } \|f\|^2 = [f, f] = \Sigma f_i^2 \\ &\text{(iii) } [Kf, f] = \sum \frac{f_i^2}{\lambda_i} \\ &\text{(iv) } [Kg, f] = \sum \frac{f_i g_i}{\lambda_i} = \sum [Kf, \phi_i] g_i \end{aligned}\right\} \quad \text{...(3.2)}$$

Let

$$f^{(n)} = f - \sum_{i=1}^{n} f_i \phi_i. \qquad \ldots(3.3)$$

Evidently,

$$[f^{(n)}, \phi_i] = 0, \; i = 1, \; 2 \; \ldots n.$$

We define a transformation of finite rank

$$K_n = \sum_{i=1}^{n} \frac{\phi_i \psi_i}{\lambda_i} \qquad \ldots(3.4)$$

where $\psi_i = G\phi_i$,

then the following results can be easily established:

$$\left.\begin{array}{l} \text{(i) } [Kf^{(n)}, f^{(n)}] = [(K - K_n)f, f] \\ \text{(ii) } [f^{(n)}, f^{(n)}] < [f, f] \\ \text{(iii) } [Kf^{(n)}, f^{(n)}] \leqslant \lambda_{n+1}^{-1} [f^{(n)}, f^{(n)}] \end{array}\right\} . \qquad \ldots(3.5)$$

Defining the norm of K in the usual manner:

$$\| K \| = \sup_{f \in H} \; [Kf, Kf]^{1/2}, \text{ such that } \| f \| = 1$$

and recalling that an operator is completely continuous if it can be approximated in norm arbitrarily closely to a linear transformation of finite rank (Riesz and Nagy 1955) we have the following:

Theorem 1—K is completely continuous with respect to the Dirichlet norm.

Since K is symmetric with respect to Dirichlet norm we have for such transformations (Riesz and Nagy 1955)

$$\sup_{\| f \| = 1} [Kf, f] = \| K \| = [Kf, Kf]^{1/2} = \sup_{\| f \| = 1} \| Kf \| \qquad \ldots(3.6)$$

$$\therefore \quad \| K - K_n \| = \sup_{\| f \| = 1} \frac{[(K - K_n)f, f]}{[f, f]}$$

$$\leqslant \sup_{\| f^{(n)} \| = 1} \frac{[Kf^{(n)}, f^{(n)}]}{[f^{(n)}, f^{(n)}]} \leqslant \lambda_{n+1}^{-1}. \qquad \ldots(3.7)$$

The last inequality has been obtained by using (3.5). Since $\lambda_n{}^{-1} \to 0$ as $n \to \infty$, therefore $\| K - K_n \| \to 0$. So that K can be approximated in norm arbitrarily closely to a linear transformation of finite rank and hence is itself completely continuous. Q.E.D.

$\lambda_n^{-1} \to 0$ is characteristic of completely continuous symmetric transformations and can be used in establishing a number of their properties and vice versa (Riesz and Nagy 1955).

4. Weakly and Strongly Convergent Sequences : Continuity of K

The following definitions (Riesz and Nagy 1955) are needed in the sequel:

(i) A sequence $\{f_n\}$ is said to tend weakly to f, i.e.

$$f_n \rightharpoonup f \text{ if } [f_n, g] \to [f, g] : g \in H.$$

(ii) A sequence $\{f_n\}$ is said to tend strongly to f, i.e. $f_n \to f$

$$\text{if } \|f_n - f\| \to 0.$$

(iii) A transformation is said to be completely continuous if it transforms a weakly convergent sequence into a strongly convergent sequence.

Evidently, every strongly convergent sequence is also weakly convergent, i.e. if $f_n \to f$, then $f_n \rightharpoonup f$.

Let $g \in H$, then since

$$|[f_n, g] - [f, g]| = |[f_n - f, g] \leqslant \|f_n - f\| \cdot \|g\| \to 0. \qquad ...(4.1)$$

$$\therefore \quad [f_n, g] \to [f, g], \text{i.e. } f_n \rightharpoonup f.$$

Thus we have the following:

Theorem 2—K is continuous and bounded in the space endowed with Dirichlet norm.

Let $\{f_n\}$ be such that $f_n \rightharpoonup f$, then since K is completely continuous $Kf_n \to Kf$ and so $Kf_n \rightharpoonup Kf$, necessarily.

If $f_n \to f$, then

$$\| Kf_n - Kf \| = \| K(f_n - f) \| \leqslant \| K \| \cdot \| f_n - f \| \to 0, \qquad ...(4.2)$$

i.e. Kf_n converges strongly to Kf. Thus K is such that it transforms weakly (strongly) convergent sequences into weakly (strongly) convergent sequences. Hence K is continuous with respect to the norm.

We may also conclude conversely that if K is continuous then it is also completely continuous. The converse may not be true in the general case, as all continuous transformations may not be completely continuous, e.g., the identity transformation (Green 1969).

The proof for the second part may be developed on identical lines by contradiction (Riesz and Nagy 1955).

Suppose that K is not bounded, then there would be a sequence $\{f_n\}$ such that

$$\| Kf_n \| > n \| f_n \| \qquad ...(4.3)$$

choosing $g_n = \frac{1}{n \| f_n \|} f_n$, we see that $\| g_n \| = \frac{1}{n} \to 0.$

Thus $g_n \to 0$. Also $Kg_n = \frac{1}{n \| f_n \|} Kf_n$. Using (4.3) we have

$$\| Kg_n \| = \frac{1}{n \| f_n \|} \| Kf_n \| > 1, \qquad \text{...(4.4)}$$

which violates Theorem 1 above and the continuity property established earlier. Hence K is bounded in norm. Thus there exists a quantity M such that for all $g \in H$

$$| Kg | \leqslant M \| g \|. \qquad \text{...(4.5)}$$

The smallest of these bounds is denoted by $\| K \|$ and defines the norm of K.

K is evidently a linear transformation since it is additive, homogeneous and bounded. We may, furthermore, show conversely, that if K is bounded, it is continuous also.

Let $f_n \to f$, then

$$\| Kf_n - Kf \| = \| K(f_n - f) \| \leqslant \| K \| \cdot \| f_n - f \| \to 0. \qquad \text{...(4.6)}$$

i.e. $\quad Kf_n \to Kf.$ Q.E.D.

5. Generalized Fourier Expansions

The set $\{\phi_i, \lambda_i^{-1}\}$, where ϕ_i are the normalized eigen-functions such that

$$[\phi_i, \phi_j] = \delta_{ij}, \qquad \delta_{ij} \begin{array}{l} = 0 \\ = 1 \end{array} \Big\} \begin{array}{l} i \neq j \\ i = j \end{array} \qquad \text{...(5.1)}$$

can now be conveniently used to give certain generalized Fourier series:

Theorem 3—If $f \in H$, then

$$\text{(i)} \qquad Kf = \sum_i \frac{f_i \phi_i}{\lambda_i} = \sum_i [Kf, \phi_i] \phi_i \qquad \text{...(5.2)}$$

$$= \sum_{i,j} [K\phi_i, \phi_j] [f, \phi_i] \phi_i$$

$$\text{(ii)} \qquad [Kf, Kf] = \sum_i \frac{f_i^2}{\lambda_i^2} \qquad \text{...(5.3)}$$

and for the transformation of finite rank n

$$K_n f = \sum_{i=1}^{n} \frac{f_i \phi_i}{\lambda_i} = \sum_{i,j}^{n} [K\phi_i, \phi_j] [f, \phi_i] \phi_i \qquad \text{...(5.4)}$$

which is a degenerate form of (i), and consequently:

$$\text{(iii)} \qquad [K_n f, f] = \sum_{i=1}^{n} \frac{f_i^2}{\lambda_i} \qquad \text{...(5.5)}$$

(iv) $$[K_n f, K_n f] = \sum_{i=1}^{n} \frac{f_i^2}{\lambda_i^2} \qquad \ldots(5.6)$$

The representation (i) above with $\lambda_i^{-1} \to 0$ is in fact a characteristic of a completely continuous transformation (Riesz and Nagy 1955).

The question of convergence of certain series including (5.5) and (5.6) may now be looked into.

Since K^3, the third iterate of K, is continuous in the ordinary sense (Kellogg 1929, p. 304), its trace defined as

$$T_3 = \int_S K^3(p, p)\, dS_p = \sum_i \frac{1}{\lambda_i^3} \qquad \ldots(5.7)$$

is a finite quantity.

Definition—If $\frac{1}{\lambda_i}$ are the eigen-values of K and $\sum \left(\frac{1}{\lambda_i}\right)^{m+1}$ converges where $m + 1$ is the smallest integer for which convergence holds, then K is said to be of genus m.

Evidently from (5.7), K is of genus two. Thus K, which is bounded in Dirichlet norm and has genus two, presents an analogue of the Hadamard's result in respect of genus of the Fredholm Denominator for bounded kernels. Moreover, since genus of K^3 is zero, we may state, as a converse of Lalesco's result that it satisfies Hölder's condition uniformly (Goursat 1964). In this respect we recall a result of Kellogg (1929, p. 300).

If $f(p)$ is continuous on S, then

$$\chi(p) = \int_S K(p, q) f(q)\, dS_q \qquad \ldots(5.8)$$

satisfies a uniform Hölder condition on S. Moreover, if F is bound of $f(p)$, there is a constant C, independent of these functions, such that $|\chi(p)| \leqslant CF$.

Now if we replace $f(p)$ in (5.8) by the second iterated kernel $K^2(p, q)$ which is known to be continuous for $p \neq q$ then

$$K^3(p, q) = \int_S K^2(p, r)\, K(r, q)\, dS_r \qquad \ldots(5.9)$$

would satisfy Hölder condition uniformly. Similarly K^4 would also satisfy it and so on. Evidently Fredholm Denominator corresponding to them will have genus zero.

In the sequel, T_m converges for $m \geqslant 3$. However, for $m = 1,\ 2$ an estimate of the order of growth may be made.

6. Order of Growth : Convergence of Series

If We recall the generalized Hölder's inequality (Phillips 1950, p. 149). $a_1, a_2, \ldots a_n$; $b_1, b_2, \ldots b_n$ are all positive and $1/\alpha + 1/\beta = 1$, then

$$\Sigma\, a_i b_i \leqslant (\Sigma\, a_i^{\alpha})^{1/\alpha} (\Sigma b_i^{\beta})^{1/\beta}. \qquad \ldots(6.1)$$

The equality sign holds only if $a_i^{\alpha} = c b_i^{\beta}$, $i = 1, 2, \ldots n$, where c is a constant.

Choosing $a_i = \dfrac{1}{\lambda_i}$, $b_i = 1$, $\alpha = 3$ and $\beta = \dfrac{3}{2}$,

$$T_1 \sim o\,(n^{2/3}) \qquad \ldots(6.2)$$

and

$$a_i = \frac{1}{\lambda_i^2},\ b_i = 1,\ \alpha = 3/2 \text{ and } \beta = 3$$

$$T_2 \sim o\,(n^{1/3}). \qquad \ldots(6.3)$$

Since $\{1/\lambda_i\}$ is an ordered set of eigen-values of K with no finite accumulation point and each eigen-value is less than unity, we have thus:

$$T_m < T_2 \sim o\,(n^{1/3}) < T_1 \sim o\,(n^{2/3}) \qquad \ldots(6.4)$$

where T_m is the trace of the mth iterate of K and is finite for $m \geqslant 3$.

Theorem 4—K_n are bounded in norm from above and below.

From (5.5) and the fact that K_n are symmetric, we have

$$\| K_n \| = \sup_{\|f\| = 1} [K_n f, f] = \sum_{i=1}^{n} \frac{f_i^2}{\lambda_i} < \frac{1}{\lambda_1}, \qquad \ldots(6.5)$$

since $\Sigma f_i^2 = 1$.

Also, from (5.6)

$$\| K_n \|^2 = \sup_{\|f\| = 1} [K_n f, K_n f] = \sum \frac{f_i^2}{\lambda_i^2}. \qquad \ldots(6.6)$$

Now choosing

$$a_i = \frac{f_i}{\lambda_i^2},\ b_i = f_i,\ \alpha = 2 \text{ and } \beta = 2, \text{ in (6.1) we get}$$

$$\| K_n \|^2 < \sqrt{T_4}. \qquad \ldots(6.7)$$

Also from (3.7) we have $\| K - K_n \| < \dfrac{1}{\lambda_{n+1}}$ whence

$$\frac{1}{\lambda_1} = \| K \| = \| K - K_n + K_n \| < \| K - K_n \| + \| K_n \| < \frac{1}{\lambda_{n+1}} + \| K_n \|.$$

Therefore.

$$\frac{1}{\lambda_1} - \frac{1}{\lambda_{n+1}} < \| K_n \| < (T_4)^{1/4} \qquad ...(6.8)$$

which gives the bounds in terms of eigen-values.

Corollary— $\sum_i \frac{f_i^2}{\lambda_i}$ and $\sum_i \frac{f_i^2}{\lambda_i^2}$ converge absolutely.

Since both the series are monotonically increasing and bounded from above they converge. Thus the series T_i, $i = 1, 2$ whose terms are weighted with $f_i^2 < 1$, converge. In fact, a stronger result holds and the series converge when the terms are weighted even with f_i.

We are now in a position to formally solve the functional equation (1.1) in terms of the eigen set $\{\phi_i, \lambda_i^{-1}\}$ on identical lines (Riesz and Nagy 1955).

Theorem 5—The functional equation

$$\phi = f + \lambda K\phi, \ f \in H$$

has the solution

$$\phi = f + \lambda \sum \frac{f_i}{\lambda_i - \lambda} \phi_i \qquad ...(6.9)$$

for $\lambda \neq \lambda_i$, $i = 1, 2 ...,$

conversely, when the series (6.9) converges, its sum ϕ is a solution of (1.1).

When $\lambda = \lambda_i, f_i$ must be zero, so as to make the coefficient of ϕ_i indeterminate. It requires that f should be orthogonal to the corresponding eigen-function in the new scalar. This fact is comparable to Fredholm's Third Theorem (Fredholm 1900) and implies that f must be orthogonal to the eigen-function of the adjoint functional equation, over S, in the usual sense.

7. Solution of the Induced Potential Problem

Functional equation similar to (1.1) arises in the case of induced potential problem considered by Howland and Vaillancourt (1966) which deals with the potential of charges induced on a smooth surface S by the external field, the medium inside and outside the surface being of different conductivity. Solution (6.9) is directly applicable.

Let $\chi(P)$ be the external field and $V(P)$ the induced potential, then the combined potential

$$U(P) = V(P) + \chi(P) \qquad ...(7.1)$$

is required to satisfy the boundary condition

$$\sigma_1 \frac{\partial U}{\partial n_+} = \sigma_2 \frac{\partial U}{\partial n_-} \qquad ...(7.2)$$

on S, where σ_1 and σ_2 are conductivities of the medium inside and outside S, n is the inward drawn normal and $+$, $-$ denote limiting values as S is approached along the normal from positive and negative sides, respectively, as determined by n. The discontinuity relation (7.2) may be written as:

$$\frac{\partial V}{\partial n_-} - \frac{\partial V}{\partial n_+} = \lambda\left(\frac{\partial V}{\partial n_-} + \frac{\partial V}{\partial n_+}\right) + 2\lambda \frac{\partial \chi}{\partial n} \qquad \ldots(7.3)$$

where $\lambda = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$. This together with the requirement that V be harmonic throughout R and R', the interior and exterior regions to S, and regular at infinity, completely determines V, the applied field $\partial\chi/\partial n$ being given.

In the physically important case $|\lambda| < 1$. When the surface S satisfies appropriate smoothness conditions (Kellogg 1929, p. 157), the potential $V(P)$, $P \in R$ or R', due to a single layer density $\mu(p)$, $p \in S$,

$$V(P) = \int_S G(P,q)\, \mu(q)\, dS_q, \quad G(P,q) = 1/2\pi r_{Pq} \qquad \ldots(7.4)$$

satisfies the discontinuity properties

$$\frac{\partial V}{\partial n_+} = -\mu + K\mu; \quad \frac{\partial V}{\partial n_-} = \mu + K\mu \qquad \ldots(7.5)$$

which, with the help of (7.3), leads to the linear integral equation of Fredholm type for density μ

$$\mu = \lambda K\mu + \lambda \frac{\partial \chi}{\partial n}. \qquad \ldots(7.6)$$

Identifying ϕ_i with μ_i and f with $\lambda \frac{\partial \chi}{\partial n}$, (6·9) gives

$$\mu = \lambda \frac{\partial \chi}{\partial n} + \lambda^2 \sum \frac{\left[\frac{\partial \chi}{\partial n}, \mu_i\right]}{\lambda_i - \lambda}\, \mu_i \qquad \ldots(7.7)$$

whence

$$V(P) = G\mu = \lambda G \frac{\partial \chi}{\partial n} + \lambda^2 \sum \frac{\left(\frac{\partial \chi}{\partial n}, V_i\right)}{\lambda_i - \lambda}\, V_i \qquad \ldots(7.8)$$

solves the induced potential problem in terms of $G\mu_i = V_i$, called Poincarè fundamental functions associated with the surface S. $G\mu_i$ are known to be the eigenfunctions of the adjoint kernel.

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