ON THE NÖRLUND SUMMABILITY OF FOURIER SERIES

by G. S. PANDEY, *Department of Mathematics, Vikram University, Ujjain*

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In the present note we improve the conditions of all the previously known theorems on Nörlund summability of Fourier series.

§1. *Definition* (Cass 1969)—A series Σa_n with sequence of partial sums $\{S_n\}$ is said to be summable to S by the Nörlund method (N, p_n^{α}), $\alpha > -1$, if

$$\lim_{n\to\infty} t_n^{(\alpha)} \to S \quad \text{as} \quad n \to \infty$$

where

$$t_n^{(\alpha)} = \frac{1}{P_n^{(\alpha)}} \sum_{\nu=0}^{n} p_{n-\nu}^{\alpha} S_{\nu}$$

$$P_n^{(\alpha)} = \sum_{\nu=0}^{n} p_{\nu}^{\alpha}$$

and $p_n > 0$ for all $n \geq 0$.

For $\alpha = 1$, this method reduces to the (N, p_n) method of summation. In case $p_n = \dfrac{1}{n+1}$, the method (N, p_n) becomes the familiar harmonic summability $\left(N, \dfrac{1}{n+1}\right)$.

Also, for

$$p_n = \binom{n+\delta-1}{\delta-1}, \quad \delta > 0,$$

the Nörlund mean (N, p_n) reduces to the (c, δ)-mean.

§2. Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over the interval $[-\pi, \pi]$. The Fourier series associated with this function is

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(2.1)$$

where a_n, b_n ; $n = 1, 2, \ldots$, are the Fourier-trigonometric coefficients of $f(x)$.

We write

$$\phi(t) \equiv \phi(x, t) = f(x + t) + f(x - t) - 2f(x).$$

§3. Iyengar (1943) proved that, if

$$\phi(t) = o\,[(\log 1/t)^{-1}] \qquad \ldots(3.1)$$

as $t \to +0$, then the series (2.1) is summable by harmonic means to $f(x)$.

Improving the above result of Iyengar, Siddiqi (1948) has shown that if

$$\Phi(t) \equiv \int_0^t |\phi(u)|\, du = o\left[\frac{t}{\log 1/t}\right] \qquad \ldots(3.2)$$

as $t \to +0$, then the series (2.1) is summable by the harmonic means to $f(x)$.

Pati (1961), on the other hand, has proved that, if

$$\Phi(t) = o\left[\frac{t}{P_{(1/t)}}\right] \qquad \ldots(3.3)$$

as $t \to +0$, then the series (2.1) is summable (N, p_n) to the value $f(x)$. Generalizing this result of Pati, Singh (1964) proved the following:

Theorem A—If (N, p_n) be a regular Nõrlund method, defined by a real, non-negative, monotonic non-increasing sequence $\{p_n\}$ such that $P_n \to \infty$, then, if

$$\Phi(t) = o\left[\frac{p_{(1/t)}}{P_{(1/t)}}\right] \qquad \ldots(3.4)$$

as $t \to +0$, the Fourier series of $f(t)$, at $t = x$, is summable (N, p_n) to $f(x)$.

The object of the present paper is to study the summability of Fourier series by (N, p_n)-method under a condition which is less stringent than the conditions of all other theorems in this line of work.

We, in fact, prove the following:

Theorem—A Nõrlund method of summation (N, p_n^{α}) defined by a real, non-negative, non-increasing sequence $\{p_n^{\alpha}\}$ such that $P_n^{\alpha} \to \infty$ as $n \to \infty$, sums the series (2.1) to $f(x)$ at every point x at which

$$\phi(t) \equiv \int_t^{\delta} \frac{|\phi(u)|}{u} P^{(\alpha)}_{(1/u)}\, du = o\,[P^{(\alpha)}_{(1/t)}],\; 0 < \delta < \pi, \text{ as } t \to +0. \qquad \ldots(3.5)$$

As a matter of fact, the natural extension of Theorem A to the case of (N, p_n^{α})-summability of the Fourier series follows, in a similar way, under the condition

$$\Phi(t) = o\left[\frac{p^{\alpha}_{(1/t)}}{P^{(\alpha)}_{(1/t)}}\right], \qquad \ldots(3.6)$$

as $t \to +0$.

But even this condition is more stringent than the condition (3.5) of our theorem. For, on integration by parts, we have

$$\phi(t) \equiv O\{[\Phi(u)\,.\,P^{(\alpha)}_{(1/u)}\,.\,u^{-1}]_t^{\delta} + \int_t^{\delta} \Phi(u)\,.\,P^{(\alpha)}_{(1/u)}\,.\,u^{-2}\,du$$

$$+ \int_t^{\delta} \Phi(u)\,.\,u^{-1}\,.\,dP^{(\alpha)}_{(1/u)}\}.$$

$$= O(1) + o\left[\frac{p^{\alpha}_{(1/t)}}{P^{(\alpha)}_{(1/t)}}\,.\,\frac{P^{(\alpha)}_{(1/t)}}{t}\right] + o\left[\int_t^{\delta} \frac{p^{\alpha}_{(1/u)}}{P^{(\alpha)}_{(1/u)}}\,.\,\frac{P^{(\alpha)}_{(1/u)}}{u^2}\,du\right]$$

$$+ o\left[\int_t^{\delta} \frac{p^{\alpha}_{(1/u)}}{P^{(\alpha)}_{(1/u)}}\,.\,\frac{1}{u}\,dP^{(\alpha)}_{(1/u)}\right]$$

$$= O(1) + o\,[P^{(\alpha)}_{(1/t)}] + o\,[\int_t^{\delta} p^{\alpha}_{(1/u)}\,.\,u^{-2}\,du]$$

$$+ o\left[\int_t^{\delta} \frac{p^{\alpha}_{(1/u)}}{P^{(\alpha)}_{(1/u)}}\,.\,\frac{1}{u}\,dP^{(\alpha)}_{(1/u)}\right]$$

$$= o\,[P^{(\alpha)}_{(1/t)}] + o\,[\int_{\delta^{-1}}^{t^{-1}} p_z^{\alpha}\,dz] + o\left[\int_{\delta^{-1}}^{t^{-1}} \frac{z\,p_z^{\alpha}}{P_z^{(\alpha)}}\,dP_z^{(\alpha)}\right]$$

$$= o\,[P^{(\alpha)}_{(1/t)}] + o\left[\sum_{k=0}^{[t^{-1}]} p_k^{\alpha}\right] + o\,[\int_{\delta^{-1}}^{t^{-1}} dP_z^{(\alpha)}]$$

$$= o(P^{(\alpha)}_{(1/t)}) \quad \text{as} \quad t \to 0. \qquad \ldots(3.7)$$

Thus we observe that the condition (3.6) ⇒ (3.5). Integrating again by parts, we have

$$\Psi(t) \equiv \int_0^t |\psi(u)|\,du, \quad \psi(u) \equiv \phi(u)\,P^{(\alpha)}_{(1/u)},$$

$$= \int_0^t -u\,\phi'(u)\,du, \quad (\phi'(u) = \frac{d}{du}\,\{\phi(u)\}).$$

$$= [-u\,\phi(u)]_0^t + \int_0^t \phi(u)\,du$$

$$= o\,[t\,P^{(\alpha)}_{(1/t)}] + o\,[\int_0^t P^{(\alpha)}_{(1/u)}\,du]$$

$$= o\,[t\,P^{(\alpha)}_{(1/t)}] + o\,[u\,P^{(\alpha)}_{(1/u)}]_0^t + o\,[\int_0^t u\,dP^{(\alpha)}_{(1/u)}]$$

$$= o\,[t\,P^{(\alpha)}_{(1/t)}]$$

from the hypothesis (3.5). Thus, we find that

$$\int_0^t | \phi(u) | P^{(\alpha)}_{(1/u)} \, du = o[t P^{(\alpha)}_{(1/t)}]. \qquad ...(3.8)$$

This implies that

$$P^{(\alpha)}_{(1/t)} \int_0^t | \phi(u) | \, du = o[t P^{(\alpha)}_{(1/t)}]$$

i.e.

$$\Phi(t) = o(t) \qquad \text{as} \quad t \to 0.$$

Hence, in view of (3.7) and (3.8), it follows that (3.5) is a weaker assertion than (3.6).

In case $\alpha = 1$, the condition (3.5) is an improvement of the condition (3.4) of Theorem A, and therefore, the improvements of the theorems of Iyengar, Siddiqi and Pati (*loc. cit.*) follow as particular cases.

§4. We shall use the following lemmas in the proof of our theorem.

Lemma 1—If $\{p_n^\alpha\}$ is a non-negative and non-increasing sequence, then for $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ and any n

$$\left| \sum_{k=a}^{b} p_k^\alpha \, e^{i(n-k)t} \right| \leq K P^{(\alpha)}_{(1/t)},$$

where K is an absolute constant.

The proof of the lemma follows on the lines of McFadden (1942).

Lemma 2—If $\{p_n^\alpha\}$ is a non-negative and non-increasing sequence, then for $1/n \leq t \leq \delta < \pi$, we have

$$| K_n(t) | \equiv \left| \sum_{k=0}^{n} p_k^\alpha \, . \, \frac{\sin (n - k + \frac{1}{2})}{\sin \frac{1}{2} t} \right| = O[t^{-1} \, . \, P^{(\alpha)}_{(1/t)}].$$

PROOF: Using Lemma 1, we obtain

$$| K_n(t) | \leq \frac{1}{\sin \frac{1}{2} t} \left| I \left(e^{\frac{1}{2} it} \sum_{k=0}^{n} p_k^\alpha \, . \, e^{i(n-k)t} \right) \right|$$

$$= O[t^{-1} \, . \, P^{(\alpha)}_{(1/t)}].$$

§5. *Proof of the Theorem*—It is well known that

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \, . \, \frac{\sin (n + \frac{1}{2}) t}{\sin \frac{1}{2} t} \, dt,$$

where $S_n(x)$ denotes the nth partial sum of the series (2.1).

Now, by the definition of (N, p_n^{α})-mean, we have

$$t_n^{(\alpha)} - f(x) = \frac{1}{P_n^{(\alpha)}} \cdot \sum_{k=0}^{n} p_k^{\alpha} S_{n-k}(x) - f(x)$$

$$= \frac{1}{P_n^{(\alpha)}} \cdot \sum_{k=0}^{n} p_k^{\alpha} [S_{n-k}(x) - f(x)]$$

$$= \frac{1}{2\pi P_n^{(\alpha)}} \cdot \int_0^{\pi} \phi(t) \cdot \sum_{k=0}^{n} p_k^{\alpha} \cdot \frac{\sin(n-k+\frac{1}{2})t}{\sin\frac{1}{2}t} dt$$

$$= \frac{1}{2\pi P_n^{(\alpha)}} \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right]$$

$$= J_1 + J_2 + J_3 \text{ (say).} \qquad \ldots(5.1)$$

By Lemma 1, we find that

$$J_1 \leq \frac{1}{2\pi P_n^{(\alpha)}} \cdot \int_0^{1/n} |\phi(t)| \cdot \sum_{k=0}^{n} p_k^{\alpha} \cdot \frac{(n-k+\frac{1}{2})t}{t/2} \cdot dt$$

$$= O\left[n \int_0^{1/n} |\phi(t)|\, dt \right]$$

$$= o(1) \text{ by } (3.8). \qquad \ldots(5.2)$$

Next using Lemma 2 and the hypothesis (3.5), we get

$$J_2 = O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} |\phi(t)| \cdot t^{-1} P_{(1/t)}^{(\alpha)} dt \right]$$

$$= O\left[\frac{1}{P_n^{\alpha}} \cdot o(P_n^{(\alpha)}) \right]$$

$$= o(1). \qquad \ldots(5.3)$$

Finally, on account of the Riemann-Lebesgue theorem, we have

$$J_3 = \frac{1}{2\pi P_n^{(\alpha)}} \int_{\delta}^{\pi} \phi(t) \cdot \sum_{k=0}^{n} p_k^{\alpha} \cdot \frac{\sin(n-k+\frac{1}{2})t}{\sin\frac{1}{2}t} \cdot dt$$

$$= o(1). \qquad \ldots(5.4)$$

This completes the proof of the theorem.

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