ON THE DECOMPOSITION OF CURVATURE TENSOR FIELDS K^i_{jhk} AND H^i_{jhk} IN RECURRENT FINSLER SPACES

by H. D. PANDE *and* H. S. SHUKLA, *Department of Mathematics, University of Gorakhpur, Gorakhpur*

(*Received* 9 *June* 1976)

Takano (1967) decomposed the curvature tensor field in a recurrent affinely connected space and studied its properties. Sinha and Singh (1970) decomposed the recurrent relative curvature tensor field and the recurrent Berwald's curvature tensor field in a Finsler space and derived certain theorems. A general decomposition of Berwald's curvature tensor field in a recurrent Finsler space was studied by Pande, and Khan (1973). The object of this paper is to decompose the curvature tensor fields K^i_{jhk} and H^i_{jhk} in recurrent Finsler spaces and to study the properties of such decompositions.

1. INTRODUCTION

We consider an n-dimensional Finsler space F_n and adopt the notations used by Rund (1959). In F_n, the covariant derivatives of a tensor field $T_j^i(x, \dot{x})$ with respect to x^k in the sense of Cartan and Berwald are given by

$$T^i_{j|k} = \partial_k^{(1)} T_j^i - \dot{\partial}_l T_j^i \Gamma^{*l}_{rk} \dot{x}^r + T_j^r \Gamma^{*i}_{rk} - T_r^i \Gamma^{*r}_{jk} \qquad ...(1.1)$$

and

$$T^i_{j(k)} = \partial_k T_j^i - \dot{\partial}_l T_j^i G^l_{rk} \dot{x}^r + T_j^r G^i_{rk} - T_r^i G^r_{jk} \qquad ...(1.2)$$

respectively. The connection coefficients $\Gamma^{*i}_{jk}(x, \dot{x})$ and $G^i_{jk}(x, \dot{x})$ are symmetric in their lower indices and are homogeneous functions of degree zero in their directional arguments.

The commutation formulae for the above two covariant derivatives are given by

$$T^i_{j|hk} - T^i_{j|kh} = T_j^r K^i_{rhk} - T_r^i K^r_{jhk} - \dot{\partial}_r T_j^i K^r_{shk} \dot{x}^s, \qquad ...(1.3)$$

and

$$T^i_{j(h)(k)} - T^i_{j(k)(h)} = T_j^r H^i_{rhk} - T_r^i H^r_{jhk} - \dot{\partial}_r T_j^i H^r_{hk}, \qquad ...(1.4)$$

---

(1) $\partial_k \equiv \partial/\partial x^k$ and $\dot{\partial}_k \equiv \partial/\partial \dot{x}^k$.

where

$$K^i{}_{jhk}(x,\dot{x}) \stackrel{\text{def}}{=} (\partial_k \Gamma^{*i}_{jh} - \dot{\partial}_l \Gamma^{*i}_{jh} \Gamma^{*l}_{rk} \dot{x}^r) - (\partial_h \Gamma^{*i}_{jk} - \dot{\partial}_l \Gamma^{*i}_{jk} \Gamma^{*l}_{rh} \dot{x}^r) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh} - \Gamma^{*i}_{mh} \Gamma^{*m}_{jk}, \quad \ldots(1.5)$$

$$H^i{}_{hjk}(x,\dot{x}) \stackrel{\text{def}}{=} \partial_k G^i{}_{hj} - \partial_j G^i{}_{hk} + G^p{}_{hj} G^i{}_{pk} - G^p{}_{hk} G^i{}_{pj} + G^i{}_{phk} \dot{\partial}_j G^p - G^i{}_{phj} \dot{\partial}_k G^p, \quad \ldots(1.6)$$

and

$$H^i{}_{jk}(x,\dot{x}) \stackrel{\text{def}}{=} \partial_k \dot{\partial}_j G^i - \partial_j \dot{\partial}_k G^i + G^i{}_{kp} \dot{\partial}_j G^p - G^i{}_{pj} \dot{\partial}_k G^p. \quad \ldots(1.7)$$

The tensor fields $H^i{}_{hjk}(x,\dot{x})$ and $H^i{}_{jk}(x,\dot{x})$ are related by

$$H^i{}_{hjk} = \dot{\partial}_h H^i{}_{jk}, \quad H^i{}_{hjk}\dot{x}^h = H^i{}_{jk}. \quad \ldots(1.8)$$

We shall use the following identities in the sequel:

$$K^i{}_{jhk} = -K^i{}_{jkh}, \quad K^i{}_{jhk} + K^i{}_{hkj} + K^i{}_{kjh} = 0, \quad \ldots(1.9)$$

$$K^r{}_{ijk|h} + K^r{}_{ikh|j} + K^r{}_{ihj|k} + (\dot{\partial}_l \Gamma^{*r}_{ij} K^l{}_{skh} + \dot{\partial}_l \Gamma^{*r}_{ih} K^l{}_{sjk} + \dot{\partial}_l \Gamma^{*r}_{ik} K^l{}_{shj})\dot{x}^s = 0, \quad \ldots(1.10)$$

$$H^i{}_{kh(j)} + H^i{}_{hj(k)} + H^i{}_{jk(h)} = 0, \quad \ldots(1.11)$$

$$H^i{}_{jkh} + H^i{}_{khj} + H^i{}_{hjk} = 0. \quad \ldots(1.12)$$

A Finsler space is said to be K-recurrent or H-recurrent according as

$$K^i{}_{jhk|l} = v_l K^i{}_{jhk}, \quad \ldots(1.13)$$

or

$$H^i{}_{jkh(l)} = \lambda_l H^i{}_{jkh}, \quad \ldots(1.14)$$

where v_l and λ_l are non-zero vector fields called the recurrent vector fields. These vector fields are homogeneous of degree zero in the $\dot{x}^i$.

Transvecting (1.14) by $\dot{x}^j$ and noting that $\dot{x}^j{}_{(l)} = 0$, we have

$$H^i{}_{kh(l)} = \lambda_l H^i{}_{kh}. \quad \ldots(1.15)$$

Contracting (1.14) with respect to i and h, we have

$$H_{jk(l)} = \lambda_l H_{jk}, \quad \ldots(1.16)$$

where

$$H_{jk} \stackrel{\text{def}}{=} H^i{}_{jki}.$$

2. Decomposition of K^i_{jhk} in a K-Recurrent Finsler Space

Let us consider the decomposition of curvature tensor field $K^i_{jhk}(x, \dot{x})$ as

$$K^i_{jhk}(x, \dot{x}) = \dot{x}^i \psi_{jhk}(x, \dot{x}), \quad ...(2.1)$$

where $\psi_{jhk}(x, \dot{x})$ is a non-zero homogeneous tensor field of degree-1 in its directional arguments and

$$\dot{x}^i v_i = \sigma. \quad ...(2.2)$$

The tensor field $\psi_{jhk}(x, \dot{x})$ is called the decomposition tensor field.

We have the following theorems:

Theorem 2.1—The decomposition tensor field $\psi_{jhk}(x, \dot{x})$ satisfies the identities:

$$(a)\ \psi_{jhk} + \psi_{jkh} = 0, \qquad (b)\ \psi_{jhk} + \psi_{hkj} + \psi_{kjh} = 0. \quad ...(2.3)$$

Proof: In view of (2.1), the identities (1.9) can be expressed as (2.3).

Theorem 2.2—In a K-recurrent Finsler space, the decomposition tensor field ψ_{jhk} behaves like a recurrent tensor field.

Proof: Differentiating (2.1) covariantly with respect to x^l and simplifying the result with the help of (1.13) and (2.1), we find

$$\psi_{jhk|l} = v_l \psi_{jhk}, \quad ...(2.4)$$

which proves the Theorem 2.2.

Theorem 2.3—In a K-recurrent Finsler space, $(v_{l|m} - v_{m|l})$ behaves like a recurrent tensor field under the decomposition (2.1).

Proof: Differentiating (2.4) covariantly with respect to x^m and using (2.4) we get

$$\psi_{jhk|lm} = v_{l|m}\psi_{jhk} + v_l v_m \psi_{jhk}. \quad ...(2.5)$$

Interchanging the indices l, m in (2.5) and subtracting the result thus obtained from (2.5) and using the commutation formula (1.3), we get

$$(v_{l|m} - v_{m|l})\,\psi_{jhk}$$

$$= -(\psi_{rhk}K^r_{jlm} + \psi_{jrk}K^r_{hlm} + \psi_{jhr}K^r_{klm} + \dot{\partial}_r\psi_{jhk}K^r_{plm}\dot{x}^p). \quad ...(2.6)$$

By virtue of (2.1) and the homogeneity property of $\psi_{jhk}(x, \dot{x})$, (2.6) yields

$$(v_{l|m} - v_{m|l})\,\psi_{jhk}$$

$$= -(\psi_{rhk}\psi_{jlm} + \psi_{jrk}\psi_{hlm} + \psi_{jhr}\psi_{klm} - \psi_{jhk}\psi_{rlm})\,\dot{x}^r. \quad ...(2.7)$$

Differentiating (2.7) covariantly with respect to x^p and using (2.4) and (2.7), we get

$$(v_{l|m} - v_{m|l})_{|p} = v_p (v_{l|m} - v_{m|l}), \qquad \text{...(2.8)}$$

which proves the Theorem 2.3.

Theorem 2.4—In a *K*-recurrent Finsler space, under the decomposition (2.1), we have

$$K^r{}_{i;k} = \frac{1}{\sigma} (\psi_{ijk} v_k + \psi_{ihk} v_j) \dot{x}^r \dot{x}^h. \qquad \text{...(2.9)}$$

PROOF: With the help of (1.13), (2.1) and the fact that $\dot{\partial}_r \Gamma^{*i}_{jk} \dot{x}^r = 0$, the Bianchi identity (1.10) takes the following form:

$$(\psi_{ijk} v_h + \psi_{ihj} v_k + \psi_{ikh} v_j) \dot{x}^r = 0. \qquad \text{...(2.10)}$$

Transvecting (2.10) by $\dot{x}^h$ and using (2.2), (2.1) and (2.3*a*), we get (2.9).

Theorem 2.5—In a *K*-recurrent Finsler space, under the decomposition (2.1), the recurrent vector field v_l satisfies the relation

$$v_l (v_{m|p} - v_{p|m}) + v_m (v_{p|l} - v_{l|p}) + v_p (v_{l|m} - v_{m|l}) = 0. \qquad \text{...(2.11)}$$

PROOF: Taking successive covariant derivative of (2.4) with respect to x^m and x^p, we have

$$\psi_{jhk|lmp} = \{v_{l|mp} + v_l (v_{m|p} + v_m v_p) + v_p v_{l|m} + v_{l|p} v_m\} \psi_{jhk}. \qquad \text{...(2.12)}$$

Commuting m and p in (2.12) and using (1.3), (2.1), (2.4) and the homogeneity property of v_l and ψ_{jhk}, we have

$$v_l \{(v_{m|p} - v_{p|m}) \psi_{jhk} + (\psi_{rhk} \psi_{jmp} + \psi_{jrk} \psi_{hmp} + \psi_{jhr} \psi_{kmp} - \psi_{jhk} \psi_{rmp}) \dot{x}^r\} = 0. \qquad \text{...(2.13)}$$

Cyclically permuting l, m, p in (2.13) and adding the three relations thus obtained and using (2.10), we have

$$\psi_{jhk} \{v_l (v_{m|p} - v_{p|m}) + v_m (v_{p|l} - v_{l|p}) + v_p (v_{l|m} - v_{m|l})\} = 0. \qquad \text{...(2.14)}$$

Since ψ_{jhk} is a non-zero tensor field, hence we have (2.11).

Theorem 2.6—In a *K*-recurrent Finsler space, under the decomposition (2.1), the decomposition tensor field ψ_{jhk} satisfies the relation

$$(\psi_{jhk|lmp} - \psi_{jhk|lpm}) + (\psi_{jhk|mpl} - \psi_{jhk|mlp}) + (\psi_{jhk|plm} - \psi_{jhk|pml}) = 0. \qquad \text{...(2.15)}$$

PROOF: From (2.12), we have

$$(\psi_{jhk|lmp} - \psi_{jhk|lpm}) = \{v_l (v_{m|p} - v_{p|m}) - v_r \dot{x}^r \psi_{lmp}\} \psi_{jhk}. \qquad \text{...(2.16)}$$

Cyclically permuting l, m, p in (2.16), we get two more relations. On adding these three relations and using (2·11) and (2.3*b*), we have (2.15).

Further considering the decomposition of the tensor field ψ_{jhk} in the form

$$\psi_{jhk} = v_j \psi_{hk}, \qquad ...(2.17)$$

where ψ_{hk} is a non-zero tensor field, we have the following theorems:

Theorem 2.7—In a *K*-recurrent Finsler space, under the decompositions (2.1) and (2.17), the tensor field $\psi_{hk}(x, \dot{x})$ behaves like a recurrent tensor field if σ is constant.

PROOF: Differentiating (2.17) covariantly with respect to x^l and using (2.4) and (2.17), we get

$$v_l v_j \psi_{hk} = v_{j|l} \psi_{hk} + v_j \psi_{hk|l}. \qquad ...(2.18)$$

Transvecting (2.18) by $\dot{x}^j$ and using (2.2) with σ constant, we get

$$\psi_{hk|l} = v_l \psi_{hk}, \qquad ...(2.19)$$

which proves Theorem 2.7.

Theorem 2.8—In a *K*-recurrent Finsler space, under the decompositions (2.1) and (2.17), the tensor field $\psi_{hk}(x, \dot{x})$ is homogeneous of degree-1 in its directional arguments.

PROOF: In view of the homogeneity property of ψ_{jhk} and v_j, the theorem follows from (2.17).

Theorem 2.9—In a *K*-recurrent Finsler space, under the decompositions (2.1) and (2.17), the necessary and sufficient condition that $v_{l|m} = v_{m|l}$ is that $\sigma = 0$.

PROOF: From (2.17) and (2.13), we have

$$v_l \{\psi_{hk}(v_{m|p} - v_{p|m}) + \psi_{mp}(v_h \psi_{rk} + v_k \psi_{hr}) \dot{x}^r\} = 0. \qquad ...(2.20)$$

With the help of (2.3*b*) and (2.17), we have

$$v_h \psi_{rk} + v_k \psi_{hr} = v_r \psi_{hk}. \qquad ...(2.21)$$

Substituting (2.21) into (2.20) and noting that the vector field v_l and the tensor field ψ_{hk} are non-zeros, we have

$$(v_{m|p} - v_{p|m}) + \sigma \psi_{mp} = 0, \qquad ...(2.22)$$

which establishes Theorem 2.9.

3. DECOMPOSITION OF H^i_{jkh} IN AN *H*-RECURRENT FINSLER SPACE

We suppose the decomposition of Berwald's curvature tensor field $H^i_{jkh}(x, \dot{x})$ in the form

$$H^i_{jkh}(x, \dot{x}) = p_j(x, \dot{x}) X^i_{kh}(x, \dot{x}), \qquad ...(3.1)$$

where $p_j(x, \dot{x})$ is a non-zero vector field and $X^i_{kh}(x, \dot{x})$ is a non-zero tensor field. Transvecting (3.1) by $\dot{x}^j$, we have

$$H^i_{kh} = p_j\dot{x}^j X^i_{kh}. \qquad \text{...(3.2)}$$

We shall prove the following theorems:

Theorem 3.1—In an H-recurrent Finsler space, under the decomposition (3.1), the recurrent vector field λ_l and the decomposition vector field p_l are proportional to each other.

PROOF: Using (3.1) in (1.12), we have

$$p_j X^i_{kh} + p_k X^i_{hj} + p_h X^i_{jk} = 0. \qquad \text{...(3.3)}$$

Applying (1.15) and (3.2) to the identity (1.11), we have

$$p_l\dot{x}^l (\lambda_j X^i_{kh} + \lambda_k X^i_{hj} + \lambda_h X^i_{jk}) = 0.$$

Since $p_l\dot{x}^l \neq 0$, therefore we have from the above equation

$$\lambda_j X^i_{kh} + \lambda_k X^i_{hj} + \lambda_h X^i_{jk} = 0. \qquad \text{...(3.4)}$$

In the light of (3.3) and (3.4), we have Theorem 3.1.

Theorem 3.2—In an H-recurrent Finsler space, under the decomposition (3.1), we have

$$\lambda_i H^i_{jkh} = \lambda_h H_{jk} - \lambda_k H_{jh}. \qquad \text{...(3.5)}$$

PROOF: Contracting (3.1) with respect to i and h, we have

$$H_{jk} = p_j X^i_{ki}. \qquad \text{...(3.6)}$$

Transvecting (3.1) by λ_i, we have

$$\lambda_i H^i_{jkh} = p_j \lambda_i X^i_{kh}. \qquad \text{...(3.7)}$$

Contracting (3.4) with respect to i and j and noting that X^i_{hk} is skew-symmetric in h and k, we have

$$\lambda_i X^i_{kh} = \lambda_h X^i_{ki} - \lambda_k X^i_{hi}. \qquad \text{...(3.8)}$$

Transvecting (3.8) by p_j and using (3.7) and (3.6), we obtain (3.5).

Theorem 3.3—In an H-recurrent Finsler space, under the decomposition (3.1), we have

$$\lambda_{i(m)} = \mu_m \lambda_i, \qquad \text{...(3.9)}$$

$$X^i_{kh(m)} = \nu_m X^i_{kh}, \qquad \text{...(3.10)}$$

for some non-zero vector fields μ_m and ν_m.

PROOF: Cyclically permuting the indices j, k, h in (3.5) and adding the three results thus obtained, we have

$$\lambda_h (H_{jk} - H_{kj}) + \lambda_j (H_{kh} - H_{hk}) + \lambda_k (H_{hj} - H_{jh}) = 0, \qquad ...(3.11)$$

where we have used the identity (1.12). Differentiating (3·11) covariantly with respect to x^m and using (1.16) and (3.11), we have

$$\lambda_{h(m)} (H_{jk} - H_{kj}) + \lambda_{j(m)} (H_{kh} - H_{hk}) + \lambda_{k(m)} (H_{hj} - H_{jh}) = 0. \qquad ...(3.12)$$

In view of (3.11) and (3.12), we have

$$\lambda_{i(m)} = \mu_m \lambda_i,$$

where μ_m is a non-zero vector field.

Differentiating (3.1) covariantly with respect to x^m and using (1.14) and (3.1), we have

$$\lambda_m p_j X^i_{kh} = p_{j(m)} X^i_{kh} + p_j X^i_{kh(m)}. \qquad ...(3.13)$$

Putting $p_j = \lambda \lambda_j$, where λ is a scalar function in (3.13), and noting that λ_j is non-zero, we have

$$X^i_{kh(m)} = \nu_m X^i_{kh},$$

where

$$\nu_m = (\lambda_m - \mu_m - \sigma_{(m)}/\sigma).$$

REFERENCES

Pande, H. D., and Khan, T. A. (1973). General decomposition of Berwald's curvature tensor fields in recurrent Finsler space. *Atti Accad. Nazionale Lincei Rend.*, **55** (6), 680–85.

Rund, H. (1959). The Differential Geometry of Finsler Spaces. Springer-Verlag, Berlin.

Sinha, B. B. (1972). Decomposition of recurrent curvature tensor fields of second order. *Prog. Math.*, **6** (1), 7–14.

Sinha, B. B., and Singh, S. P. (1970). On decomposition of recurrent curvature tensor fields in Finsler spaces. *Bull. Calcutta math. Soc.*, **62**, 91–96.

Takano, K. (1967). Decomposition of curvature tensor in a recurrent space. *Tensor, N.S.*, **18** (3), 343–47.