TORSIONAL VIBRATIONS OF SPHERICAL SHELLS

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The single equation governing torsional vibrations of an anisotropic spherical shell is solved in terms of associated Legendre function $P_n^1(\cos\theta)$ and Bessel function $J_m(\omega r)$. Numerical results are obtained for the natural frequencies of the shells of various opening angles for clamped and free edge conditions. The effect of the thickness of the shell on the frequency is also studied. Three different material shells have been taken for the purpose.

1. INTRODUCTION

Naghdi and Kalnins (1962) gave the general solution for vibration of spherical shells. Prasad (1964) solved the problem of non-symmetric vibrations of non-shallow isotropic spherical shells taking the secondary effects into account. Krishna (1965) gave the numerical work for free and clamped edge conditions. He solved the problems of torsional and non-torsional axisymmetric vibrations separately. Jain (1966) discussed the torsional vibrations of isotropic spherical shells solving the equation of motion by the method of separation of variables. Krishna (1967) took the displacement component to be a linear function of the radial coordinate and then solved the problem on torsional vibrations of transversely isotropic spherical shell. In the present paper we have considered the material to be anisotropic, taking both the elastic constants needed in the equation of motion of torsional vibrations as independent and solved the problem. Extensive numerical work has been done for three different material shells.

2. BASIC EQUATIONS

The shell is referred to a system of spherical coordinates. (r', θ, ϕ), where the centre of the shell is taken as the origin and its axis of rotational symmetry as the line $\theta = 0$. R is the radius and h' the uniform thickness. We shall use the terms r, h, such that $(r, h) = (r', h')/R$. So the outer and inner faces of the shell are $r = 1 \pm h/2$. $\theta = \theta_0$, $\theta = \theta_1$ $(\theta_0 > \theta_1)$ are the two edges of the shell.

Here we confine to the torsional vibrations of the shell. So we take only the displacement u in ϕ-direction. We further see that the motion is axisymmetric.

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So all the relevant quantities are independent of ϕ. The non-zero strain components are

$$e_{\phi r} = \frac{\partial u}{\partial r} - \frac{u}{r}$$

$$e_{\theta\phi} = \frac{1}{r}\left(\frac{\partial u}{\partial \theta} - u \cot \theta\right). \qquad \ldots(2\cdot 1)$$

The stress-strain relations for the torsional deformation of an orthotropic shell are

$$\tau_{\phi r} = c_{44} e_{\phi r}$$

$$\tau_{\theta\phi} = c_{66} e_{\theta\phi}. \qquad \ldots(2\cdot 2)$$

Here c_{44}, c_{66} are the two independent elastic constants involved in the torsional vibrations.

For the present case two stress equations out of three become identities and the remaining equation is

$$\frac{\partial \tau_{\phi r}}{\partial r}\tau_{\phi r} + \frac{\partial \tau_{\theta\phi}}{\partial \theta}\tau_{\theta\phi} + \frac{1}{r}(3\tau_{\phi r} + 2\tau_{\theta\phi}\cot\theta) = \rho R^2 \frac{\partial^2 u}{\partial t^2}. \qquad \ldots(2\cdot 3)$$

Here ρ is the density of the shell. The use of (2·1), (2·2) in (2·3) gives

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} - 2u + \frac{c_{66}}{c_{44}}\left[\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\frac{\partial u}{\partial \theta} + (1 - \cot^2\theta)\, u\right]$$

$$= \rho\frac{R^2}{c_{44}} r^2 \frac{\partial^2 u}{\partial t^2}. \qquad \ldots(2\cdot 4)$$

3. Solution of Equation of Motion

We assume a separable solution of (2·4) in the form

$$u = u_1(r)\, u_2(\theta) \exp(ipt). \qquad \ldots(3\cdot 1)$$

Here u_1 is a function of r alone, u_2 a function of θ only and p the circular frequency. On substituting from (3·1) in (2·4), we get

$$\frac{1}{u_1}\left(r^2\frac{\partial^2 u_1}{\partial r^2} + 2r\frac{\partial u_1}{\partial r}\right) - 2 + \omega^2 r^2$$

$$= -\frac{c_{66}}{c_{44}}\frac{1}{u_2}\left(\frac{\partial^2 u_2}{\partial \theta^2} + \cot\theta\frac{\partial u_2}{\partial \theta}\right) - \frac{c_{66}}{c_{44}}(1 - \cot^2\theta), \qquad \ldots(3\cdot 2)$$

where

$$c_{44}\omega^2 = \rho R^2 p^2. \qquad \ldots(3\cdot 3)$$

The terms in the left-hand side of (3·2) are functions of r alone and those in the right-hand side are functions of θ alone. If β is the constant of separation, then

$$\frac{\partial^2 u_1}{\partial r^2} + \frac{2}{r}\frac{\partial u_1}{\partial r} + \left(\omega^2 - \frac{2+\beta}{r^2}\right) u_1 = 0. \qquad \ldots(3\cdot4)$$

And

$$\frac{\partial^2 u_2}{\partial \theta^2} + \cot\theta \frac{\partial u_2}{\partial \theta} + (2 - \operatorname{cosec}^2\theta)\, u_2 + \frac{c_{44}}{c_{66}} \beta\, u_2 = 0. \qquad \ldots(3\cdot5)$$

The solution of eqn. (3·4) is given by

$$u_1 = \frac{1}{\sqrt{r}} [A_1 J_m(\omega r) + A_2 J_{-m}(\omega r)], \qquad \ldots(3\cdot6)$$

where A_1, A_2 are arbitrary constants. Here J_m, J_{-m} are Bessel functions. J_{-m} is independent of J_m as m is not an integer. Also

$$m^2 = 0\cdot25 + 2 + \beta. \qquad \ldots(3\cdot7)$$

Also the solution of (3·5) is given by

$$u_2 = B_1 P_n^1(\cos\theta) + B_2 Q_n^1(\cos\theta), \qquad \ldots(3\cdot8)$$

where

$$n(n+1) = 2 + \beta\, c_{44}/c_{66}, \qquad \ldots(3\cdot9)$$

B_1, B_2 are arbitrary constants and P_n^1, Q_n^1 are associated Legendre functions of the first and second kinds respectively.

Hence the complete solution of (2.4) is given by

$$u = \frac{1}{\sqrt{r}} [A_1 J_m(\omega r) + A_2 J_{-m}(\omega r)] \times [B_1 P_n^1(\cos\theta) + B_2 Q_n^1(\cos\theta)] \times \exp(ipt). \qquad \ldots(3\cdot10)$$

We take the shell to be closed at the apex. So we set $B_2 = 0$. Then the solution is

$$u = \frac{1}{\sqrt{r}} [A_1' J_m(\omega r) + A_2' J_{-m}(\omega r)] P_n^1(\cos\theta) \exp(ipt). \qquad \ldots(3\cdot11)$$

Here A_1', A_2' are the new arbitrary constants.

4. Edge Conditions

Clamped Edge Condition

For a clamped edge the displacement u is zero at the edge $\theta = \theta_0$, i.e.,

$$P_n^1(\cos\theta_0) = 0. \qquad \ldots(4\cdot1)$$

Equation (4·1) gives us the values of n for each given value of θ_0. The different values of n correspond to various modes of torsional vibrations of a shell with clamped edge.

Free Edge Condition

The edge condition for a free edge is

$$\tau_{\theta\phi} = 0 \text{ at } \theta = \theta_0.$$

This leads to

$$\frac{\partial}{\partial\theta}[P_n^1(\cos\theta)] - \cot\theta\, P_n^1(\cos\theta) = 0 \qquad \text{...(4·2)}$$

at $\theta = \theta_0$. Equation (4·2) gives the values of n for the various normal modes of torsional vibrations corresponding to each given value of the opening angle θ_0.

5. Frequency Equation

The boundaries of the shell are

$$r = 1 + h/2 = a_1 \text{ (say)}$$

$$r = 1 - h/2 = a_2 \text{ (say)}.$$

Here the boundaries are taken to be free from tractions. So we have

$$\tau_{\phi r} = 0 \qquad \text{...(5·1)}$$

at $r = a_1$ and at $r = a_2$. Use of (5·1), (3·11) gives two equations for the two boundaries, namely

$$\sum_{j=1}^{2} A_j' B_{ij} = 0, \; i = 1, 2. \qquad \text{...(5·2)}$$

Here

$$B_{ij} = \gamma_j J_{m_j}(\omega r) + a_i \omega J_{m_j-1}(\omega r),$$

where

$$m_1 = m, m_2 = -m, \gamma_j = -m_j - 3/2.$$

Now eliminating A_j' from (5·2) we get the frequency equation

$$|B_{ij}| = 0. \qquad \text{...(5·3)}$$

6. Numerical Results

Numerical results for the frequencies for the torsional vibrations of the spherical shells for various values of the opening angle θ_0 for clamped and free edge conditions have been computed. For the known frequency, eqn. (3·11) gives the displacement u. Three arbitrary sets of values of elastic constants have been taken for the purpose. Figure 1 shows the values of natural frequencies ω of torsional vibrations with clamped edge condition for first four modes for various values of θ_0 for the shells made of three different materials. Figure 2 shows the same for the free edge condition for first normal mode only. Figure 3 shows the variation of the frequencies of the lowest mode of a hemispherical shell made of material I with the thickness.

The study of Figs. 1 and 2 reveals that the frequency increases as the opening angle decreases. The variations are larger for free edge condition than for clamped edge condition. Figure 3 gives that the frequency decreases with the increase in the thickness. The variations are larger for free edge than for clamped.

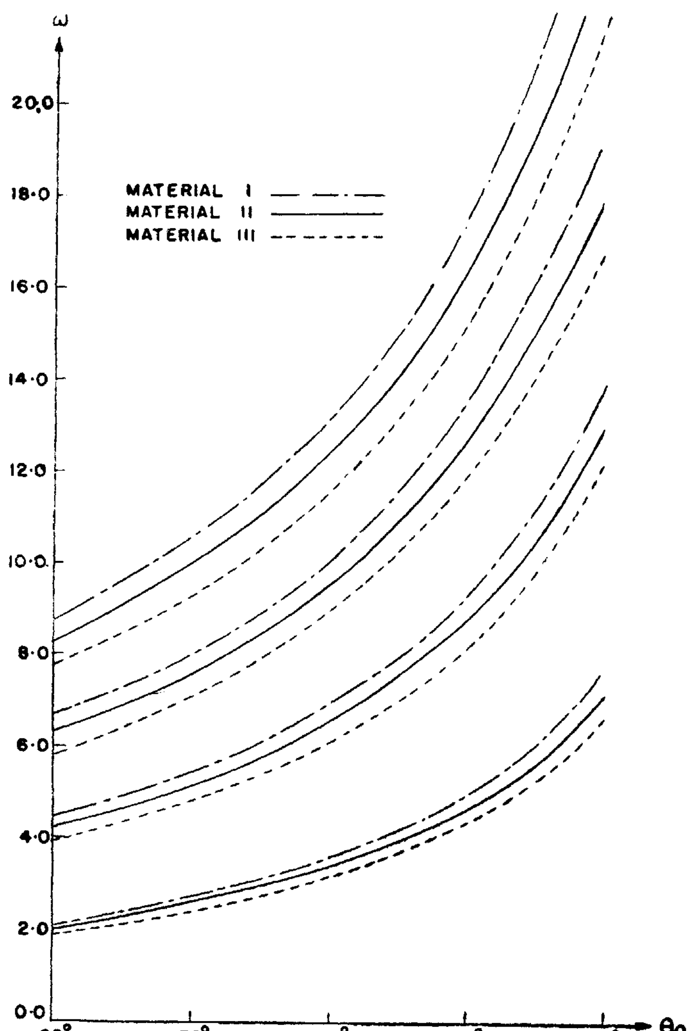

FIG. 1. Frequencies of first four modes of vibrations for various opening angles θ_0 with clamped edge condition.

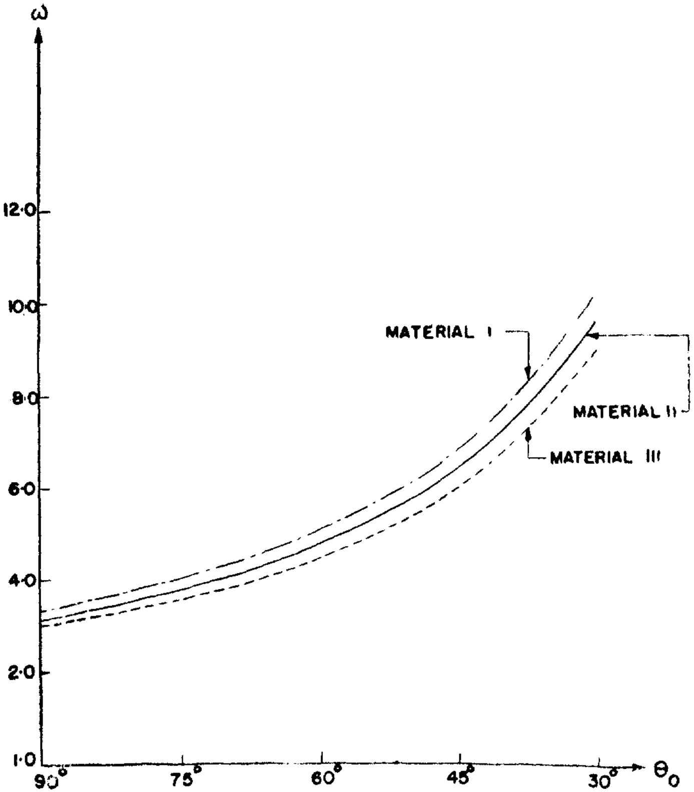

FIG. 2. Frequencies of first mode of vibration for various θ_0 with free edge condition.

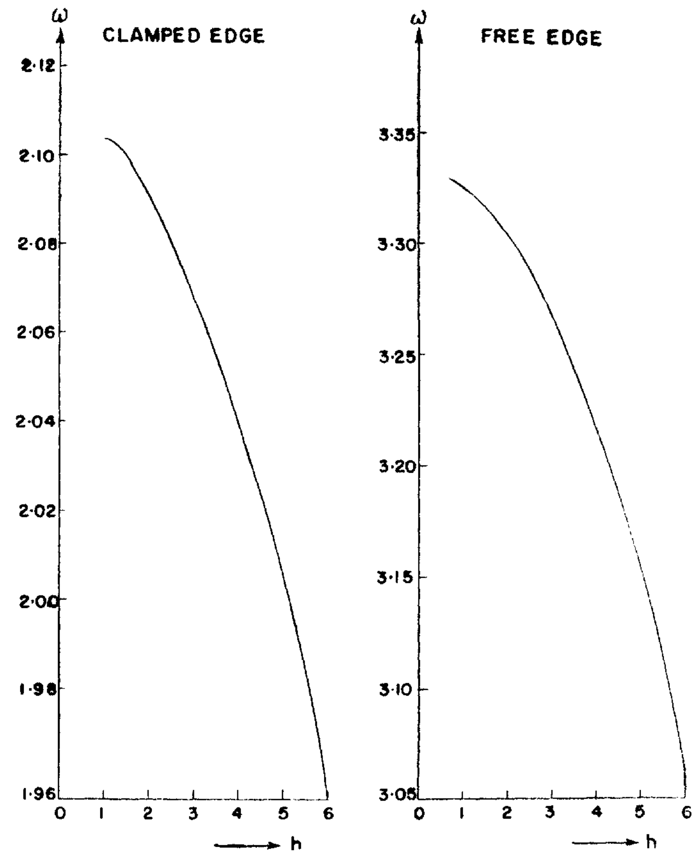

FIG. 3. Variations in frequency of the first mode of vibration of a hemispherical shell with the thickness h.

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