BENDING OF A THIN CIRCULAR PLATE SUBJECT TO PARABOLIC LOADING OVER A CONCENTRIC ELLIPTIC LIMACON

by F. R. Barsoum, *Faculty of Engineering, Mathematical Physics Department, Alexandria University, Alexandria, Egypt*

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Treated is the bending of a thin circular plate acted upon normally by parabolic loading distributed over the area of a concentric elliptic limacon, and subject to boundary conditions covering the usual rigidly clamped and simply supported boundaries. The solution is expressed in exact finite terms within the framework of the classical theory of thin elastic plates. Known solution appears as a limiting case of the results given here.

Introduction

Methods using complex variables have been applied to plate problems (refer Sokolnikoff 1942 and Muskhelishvili 1949). Tiffen (1955) dealt with clamped slabs subject to isolated interior loadings; the slabs occupying half-planes, regions which can conformally mapped on half-planes and infinite strips. Muskhelishvili's method was used by Gray (1952), Deverall (1957) and Yu (1956) in the analysis of problems concerning plates under isolated or continuous loadings. Ghose (1925) worked out the problem of a clamped circular plate subject to uniform load distributed over the sectorial area bounded by two concentric circular area and two radii. A systematic and rather comprehensive treatment on the bending of the circular plate was given by Shin-Min Jen (1949).

A general form of the boundary condition defining certain types of constraints at the boundary of a thin plate has been introduced by Bassali and Dawoud (1956 *a*, *b*). This assumes the constancy of the ratio between the boundary bending moments and has the advantage of including the usual clamped and simply supported boundaries as well as the other special cases. The introduction of this boundary condition is of practical importance since neither rigidly clamped nor simply supported conditions can be realized exactly under actual physical conditions, and thus any case met in practice must lie somewhere between these two limiting cases. The bending of thin circular plates subject to the general boundary conditions and bent with various types of loadings contributed over the area of a concentric ellipse have been studied by various authors (Bassali and Nassif 1959, Bassali 1959, 1960, 1965).

In the present paper we obtain the solution for the bending of a thin circular plate bent by a parabolic loading distributed over the area of a concentric elliptic limacon and subject to the general boundary condition. In the limiting case when the loaded area becomes a concentric circle, results agree with those obtained before (Bassali 1960).

1. Mathematical Formulation of the Problem

Let C denote the boundary $x^2 + y^2 = c^2$ of a thin elastic circular plate with centre O, divided into two regions 1 and 2 by the contour Γ of the elliptic limacon whose parametric equations are

$$\left.\begin{aligned} x &= a(\cos\phi + b\cos 2(\phi)) \\ y &= a(\sin\phi + b\sin 2\phi) \end{aligned}\right\} a > 0,\ 0 < b \leqslant \tfrac{1}{2} \qquad \ldots(1\cdot 1)$$

where region 1 lies inside Γ and region 2 between Γ and C (see Fig. 1). We assume that the plate is subject to the normal pressure intensities

$$p_1 = p_0(x^2 + y^2) = p_0 z\bar{z} \qquad \ldots(1\cdot 2a)$$

$$p_2 = 0 \qquad \ldots(1\cdot 2b)$$

where p_0 is a constant, $z = x + iy$ and bars are used to denote conjugate quantities.

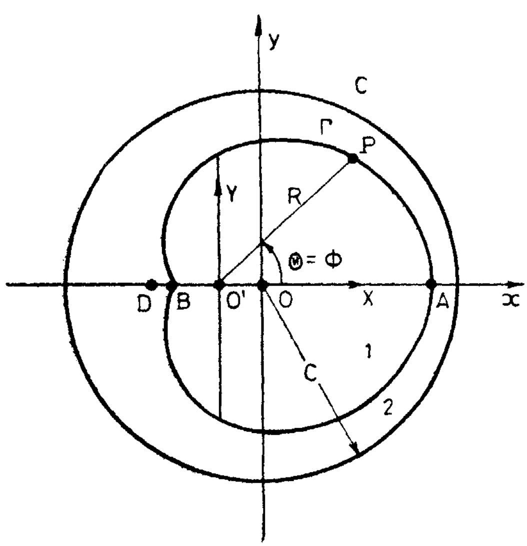

Fig. 1

According to the classical small bending theory of thin plates the deflection, measured positively upwards, at any point $z = x + iy = re^{i\theta}$ of the plate is given by

$$w_j = \bar{z}\phi_j(z) + z\bar{\phi}_j(\bar{z}) + \psi_j(z) + \bar{\psi}_j(\bar{z}) + W_j(z, \bar{z})\,(j = 1, 2) \qquad \ldots(1\cdot 3)$$

where $\phi_j(z)$, $\psi_j(z)$ are the complex potentials which are regular in their domains and $W_j(z,\bar{z})$ $(j=1, 2)$ are the particular integrals of the partial differentials equations

$$\nabla^4 w_j = 16\ \partial^4 w_j/\partial z^2\,\partial\bar{z}^2 = -\,p_j(z,\bar{z})/D \quad (j=1,\ 2) \qquad ...(1.4)$$

D is the flexural rigidity of the plate.

The transition conditions to which w_1 and w_2 must conform along Γ expressing the continuity of the deflections, slopes, moments and shears may be written as

$$\Big[w\Big]_2^1 = \Big[\frac{\partial w}{\partial z}\Big]_2^1 = \Big[\frac{\partial^2 w}{\partial z\partial\bar{z}}\Big]_2^1 = \Big[\frac{\partial^3 w}{\partial z^2\,\partial\bar{z}}\Big]_2^1 = 0. \qquad ...(1.5)$$

Adopting Stevenson's method of specification of stress (Stevenson 1943), we take as the boundary conditions along C

$$w = 0,\ \overleftarrow{r\theta}/\overleftarrow{\theta r} = (B+\lambda)/(B-\lambda)\ (1 \geqslant \lambda \geqslant -B), \qquad ...(1.6)$$

where λ is a dimensionless constant, $B=(1-\eta)/(1+\eta)$ and η is Poisson's ratio for the material of the plate. Following Bassali and Dawoud (1956*a*), the conditions (1.6) easily lead to

$$w = 0,\ c\,\partial^2 w/\partial r^2 + v\partial w/\partial r = 0, \qquad ...(1.7a)$$

or

$$\text{Re}\,[\bar{z}\,\phi_2(z) + \psi_2(z)] = 0, \qquad ...(1.7b)$$

$$\text{Re}\ [2\phi_2{}' - \lambda\,(z\phi_2{}'' + z^2\psi_2{}''/c^2)] = 0, \qquad ...(1.7c)$$

along C, where

$$v = (1+\lambda)/(1-\lambda)\ \ (\infty \geqslant v \geqslant \eta), \qquad ...(1.8)$$

Re stands for the real part and primes on ϕ, ψ denote differentiation with respect to z. The boundary C is clamped or simply supported according as $\lambda = 1$ $(v=\infty)$ or $\lambda = -B\,(v=\eta)$, respectively.

The problem to be solved now is the determination of the particular integral $W_1(z,\ \bar{z})$ and the regular functions $\phi_1(z)$, $\phi_2(z)$, $\psi_1(z)$ and $\psi_2(z)$ which satisfy the continuity requirements (1.5) and the boundary conditions (1.7*b*) and (1.7*c*).

2. Mapping Function and Solution of the Transition Conditions

To facilitate the solution of the transition conditions we use conformal mapping of the loaded region onto another region occupied by a unit circle. It can be easily seen that the mapping function

$$z = x + iy = a\,\xi\,(1 + b\zeta)\ \ a > 0, \qquad ...(2.1)$$

transforms the interior of Γ given by (1.1) in the z-plane on the interior of the unit circle $\zeta = e^{i\phi}$ in the ζ-plane. Noting that the mapping function (2.1) ceases to be conformal at $z'(\zeta)=0$ which gives $\zeta=-1/(2b)$, thus the mapping is conformal for $0\leqslant b\leqslant\frac{1}{2}$ and there will be no critical points inside $|\zeta|=1$.

To show that eqn. (1.1) represents an elliptic limacon, let A and B be the intersections of Γ with the x-axis corresponding to $\phi=0$ and $\phi=\pi$, respectively. From (1.1) we see that

$$OA=a(1+b),\quad OB=a(1-b).$$

Transfer the origin O to the point $O'(-ab, 0)$ such that any point P on Γ has the Cartesian coordinates (X, Y) and the polar coordinates (R, Θ) referred to the Cartesian axes $O'x$, $O'Y$ and O' as pole, $O'x$ as initial line: then

$$X=x+ab=a\cos\phi\,(1+2b\cos\phi),$$

$$Y=y=a\ \sin\ \phi\,(1+2b\cos\phi),$$

$$R=a(1+2b\cos\phi),\quad Y/X=\tan\ \Theta=\tan\phi.$$

Hence $\Theta=\phi$ and the polar equation of Γ referred to O' as pole is

$$R=a(1+2b\cos\Theta). \qquad ...(2.2)$$

Equation (2.2) is the inverse of the ellipse $R=L/(1+\varepsilon\cos\Theta)$ with respect to the circle of inversion $R=k$ whose centre is O' and where $L=k^2/a$, $\varepsilon=2b$. For this reason eqn. (2.2) of Γ represents an elliptic limacon. When $b=\frac{1}{2}$ the limacon becomes a cardoid.

From (1.2) and (1.4) the particular integrals may be taken as

$$W_1=-Kz^3\bar{z}^3,\quad W_2=0, \qquad ...(2.3)$$

where $K=p_0/(384\,D)$. ...(2.4)

Substituting from (2.3) in (1.3), introducing the resulting expressions in the continuity conditions (1.5), using (2.1) and noticing that along Γ we have $\zeta=\zeta^{-1}$, we find

$$[\bar{z}\,\phi(z)+z\,\bar{\phi}(\bar{z})+\psi(z)+\bar{\psi}(\bar{z})]_2^1=Ka^6\,[1+9b^2+9b^4+b^6$$
$$+3b(1+3b^2+b^4)(\zeta+\zeta^{-1})+3b^2(1+b^2)(\zeta^2+\zeta^{-2})$$
$$+b^3(\zeta^3+\zeta^{-3})], \qquad ...(2.5a)$$

$$[\bar{z}\phi'(z)+\bar{\phi}(\bar{z})+\psi'(z)]_2^1=3Ka^5\,[b(2+3b^2)+b^2\zeta$$
$$+(1+6b^2+3b^4)\zeta^{-1}+b(3+6b^2+b^4)\zeta^{-2}$$
$$+b^2(3+2b^2)\zeta^{-3}+b^3\zeta^{-4}], \qquad ...(2.5b)$$

$$[\phi'(z)+\bar{\phi}'(\bar{z})]_2^1=qKa^4\,[1+4b^2+b^4+2b(1+b^2)(\zeta+\zeta^{-1})$$
$$+b^2(\zeta^2+\zeta^{-2})], \qquad ...(2.5c)$$

$$[\phi''(z)]_2^1=18\,Ka^3\,[b+(1+2b^2)\zeta^{-1}+b(2+b^2)\zeta^{-2}+b^2\zeta^{-3}]. \qquad ...(2.5d)$$

From (2.1) we have

$$\frac{d}{dz} = \frac{1}{a(1 + 2b\zeta)} \frac{d}{d\zeta}. \qquad \text{...(2.6)}$$

By integration and using (2.6), it can be easily shown that the continuity conditions (2.5) yield

$$[\phi(z)]_2^1 = 18\,Ka^5 \Big[\frac{b^2}{2}\zeta^{-1} - \left(\frac{3}{4} + 9b^2 + \frac{31}{4}b^4\right)\zeta + \frac{b}{4}(5 - 3b^4)\zeta^2$$

$$+ \frac{b^2}{3}(7 + 8b^2)\zeta^3 + \frac{b^3}{2}\zeta^4 + \Big\{\frac{z}{a}(1 + 6b^2 + 2b^4)$$

$$- 2b(1 + 2b^2)\Big\} \log \zeta \Big], \qquad \text{...(2.7}a\text{)}$$

$$[\psi(z)]_2^1 = 18\,Ka^6 \Big[\frac{7}{9} + 8b^2 + 8b^4 + \frac{7}{9}b^6 + b\left(\frac{2}{3} + \frac{23}{2}b^2 + 8b^4\right)\zeta$$

$$- b^2\left(\frac{7}{3} + \frac{5}{2}b^2\right)\zeta^2 - \frac{8}{9}b^3\zeta^3 - b\left(1 + \frac{13}{3}b^2 + 2b^4\right)\zeta^{-1}$$

$$- b^2\left(\frac{1}{3} + \frac{b^2}{2}\right)\zeta^{-2} - \frac{b^3}{18}\zeta^{-3} + \Big\{\frac{2}{3} + 8b^2 + 10b^4 + \frac{4}{3}b^6$$

$$- 2b(1 + 2b^2)\frac{z}{a}\Big\} \log \zeta \Big]. \qquad \text{...(2.7}b\text{)}$$

As a check, letting b equal to zero, the mapping function (2.1) yields $z = a\zeta$ and (2.7*a*), (2.7*b*) reduce to

$$\Big[\phi(z)\Big]_2^1 = \frac{p_0}{32\,D}a^4 z\left(\log\frac{z}{a} - \frac{3}{4}\right), \qquad \text{...(2.8}a\text{)}$$

$$\Big[\psi(z)\Big]_2^1 = \frac{p_0}{48\,D}a^6\left(\log\frac{z}{a} + \frac{7}{6}\right), \qquad \text{...(2.8}b\text{)}$$

which are in agreement with eqns. (4.30) and (4.31) of (2).

From (2.1), the point ζ corresponding to the point z on Γ is given by

$$\zeta = (Z - 1)/(2b),\ \zeta^{-1} = a(Z + 1)/(2z), \qquad \text{...(2.9)}$$

where

$$Z = (1 + 4b\,z/a)^{1/2}. \qquad \text{...(2.10)}$$

Substituting for ζ, ζ^{-1} from (2.9) in (2.7 *a*) and (2.7*b*), we get after some algebraic manipulation

$$\Big[\phi(z)\Big]_2^1 = 18Ka^5\Big[F_1(z) + ZF_2(z) + \frac{b^2}{4}\frac{a}{z} + \Big\{2b(1 + 2b^2)$$

$$- (1 + 6b^2 + 2b^4)\frac{z}{a}\Big\}\log\frac{a(Z + 1)}{2z}\Big], \qquad \text{...(2.11}a\text{)}$$

$$\left[\psi(z)\right]_2^1 = 18\,Ka^6\left[F_3(z) + ZF_4(z) - b\left(\frac{1}{2} + \frac{5}{2}b^2 + \frac{3}{2}\,b^4\right)\frac{a}{z}\right.$$
$$- \frac{b^2}{6}\,(1 + 2b^2)\,\frac{a^2}{z^2} - \frac{b^3}{36}\,\frac{a^3}{z^3} + \left\{2b\,(1 + 2b^2)\right.$$
$$\left.\left. - \left(\frac{2}{3} + 8b^2 + 10\,b^4 + \frac{4}{3}\,b^6\right\}\log\frac{a\,(Z + 1)}{2z}\right], \qquad \text{...(2.11}b\text{)}$$

where

$$F_1(z) = \frac{1}{12}\,\frac{1}{b} + \frac{19}{6}\,b + \frac{7}{3}\,b^3 - \left(\frac{5}{4} + 4b^2 + \frac{3}{4}\,b^4\right)\frac{z}{a} + \frac{b}{2a^2}\,z^2, \qquad \text{...(2.12}a\text{)}$$

$$F_2(z) = -\left(\frac{1}{12}\,\frac{1}{b} + \frac{19}{6}\,b + \frac{7}{2}\,b^3\right) + \frac{2}{3}\,(1 + 2b^2)\,\frac{z}{a} + \frac{ab^2}{4}\,\frac{1}{z}, \qquad \text{...(2.12}b\text{)}$$

$$F_3(z) = -\frac{5}{18} + b^2 + 4b^4 + \frac{7}{9}\,b^6 - b\left(1 + \frac{5}{2}\,b^2\right)\frac{z}{a}, \qquad \text{...(2.12}c\text{)}$$

$$F_4(z) = \frac{19}{18} + 7b^2 + 4b^4 - \frac{4}{9}\,\frac{b}{a}\,z - b\left(\frac{1}{2} + \frac{13}{6}\,b^2 + b^4\right)\frac{a}{z}$$
$$- \frac{b^2}{18}\,(3 + 5b^2)\,\frac{a^2}{z^2} - \frac{b^3}{36}\,\frac{a^3}{z^3}. \qquad \text{...(2.12}d\text{)}$$

3. Determination of the Potential Functions

The expression (2.10) for Z has two branches and the branch point is at $z = -\,a/(4b)$. In Fig. 1, let D represent the point $z = -\,a/(4\,b)$ where $a > 0$ $0 \leqslant b \leqslant \frac{1}{2}$. To show that D lies outside or on the contour Γ: we have $OD = a/(4b)$,

$$OB = a\,(1 - b),\; OD - OB = a\,(1 - 4b + 4b^2)/(4b) = a\,(1 - 2b)^2/(4b)$$
$$\geqslant 0,$$

for $a > 0$, $0 \leqslant b \leqslant \frac{1}{2}$ and this proves that D does not lie inside Γ. Hence Z is uniform in region 1 and we infer that the terms containing Z occur in $\phi_1(z)$ and $\psi_1(z)$. The terms containing negative powers of z have a pole at $z = 0$ and therefore these terms do not appear in $\phi_1(z)$ and $\psi_1(z)$. We have also to exclude from $\phi_1(z)$ and $\psi_1(z)$ the singular parts of $ZF_2(z)$ and $ZF_4(z)$, respectively. Since we require w_2 to vanish on C, it follows that the terms with negative powers of z which appear in $\phi_2(z)$ and $\psi_2(z)$ have to be associated with other terms satisfying the condition (1.7b). From (2.11a), (2.11b) and carrying out the foregoing procedure we can determine that the appropriate forms of the potential functions may be taken as

$$\phi_1(z) = 18\,Ka^5\Big[azG(z) + \Big\{2b(1 + 2b^2) - (1 + 6b^2 + 2b^4)\frac{z}{a}\Big\}$$

$$\times \log\frac{a(Z+1)}{2c} + F_1(z) + ZF_2(z) - \frac{b^2}{4}\frac{a}{z} + \frac{ab^2}{2c^4}z^3\Big], \qquad \ldots(3\cdot 1a)$$

$$\psi_1(z) = 18Ka^6\Big[-c^2 G(z) + \Big\{2b(1 + 2b^2)\frac{z}{a}$$

$$-\Big(\frac{2}{3} + 8b^2 + 10b^4 + \frac{4}{3}b^6\Big)\Big\}\log\frac{a(Z+1)}{2c}$$

$$+ F_3(z) + ZF_4(z) + \frac{ab}{2}(1 + 5b^2 + 3b^4)\Big(\frac{1}{z} - \frac{2z}{c^2}\Big)$$

$$+ \frac{a^2b^2}{6}(1 + 2b^2)\Big(\frac{1}{z^2} - \frac{2z^2}{c^4}\Big) + \frac{a^3b^3}{36}\Big(\frac{1}{z^3} - \frac{2z^3}{c^6}\Big)\Big], \qquad \ldots(3.1b)$$

$$\phi_2(z) = 18Ka^5\Big[azG(z) + \Big\{2\frac{b}{a}(1 + 2b^2) - (1 + 6b^2 + 2b^4)\frac{z}{a}\Big\}\log\frac{z}{c}$$

$$-\frac{ab^2}{2}\Big(\frac{1}{z} - \frac{z^3}{c^4}\Big)\Big], \qquad \ldots(3.2a)$$

$$\psi_2(z) = 18Ka^6\Big[-c^2 G(z) + \Big\{2b(1 + 2b^2)\frac{z}{a}$$

$$-\Big(\frac{2}{3} + 8b^2 + 10b^4 + \frac{4}{3}b^6\Big)\Big\}\log\frac{z}{c} + ab(1 + 5b^2 + 3b^4)$$

$$\times\Big(\frac{1}{z} - \frac{z}{c^2}\Big) + \frac{a^2b^2}{3}(1 + 2b^2)\Big(\frac{1}{z^2} - \frac{z^2}{c^4}\Big)$$

$$+ \frac{a^3b^3}{18}\Big(\frac{1}{z^3} - \frac{z^3}{c^6}\Big)\Big], \qquad \ldots(3.2b)$$

where $G(z)$ is a regular function all over the plate. As a check, substituting from (3.2*a*), (3.2*b*) and (2.3) in (1.3) we obtain

$$w_2 = 36Ka^6\Big[(r^2 - c^2)\ \mathrm{Re}\ G(z) - \Big\{\frac{2}{3} + 8b^2 + 10b^4 + \frac{4}{3}b^6$$

$$+ (1 + 6b^2 + 2b^4)\frac{r^2}{\alpha^2}\Big\}\log\frac{r}{c} + \Big\{\frac{4b}{a}(1 + 2b^2)\log\frac{r}{c}$$

$$+ ab(1 + 5b^2 + 3b^4)\Big(\frac{1}{r} - \frac{r}{c^2}\Big) + \frac{a^2b^2}{3}(1 + 2b^2)\Big(\frac{1}{r^2} - \frac{r^2}{c^4}\Big)\Big\}\cos\theta$$

(equation continued on p. 444)

$$+ \left\{\frac{a^2b^2}{3}(1 + 2b^2)\left(\frac{1}{r^2} - \frac{r^2}{c^4}\right) - \frac{b^2}{2}\left(1 - \frac{r^4}{c^4}\right)\right\} \cos 2\theta$$

$$+ \frac{a^3 b^3}{18}\left(\frac{1}{r^3} - \frac{r^3}{c^6}\right)\cos 3\theta \Bigg], \qquad \text{...(3.3)}$$

which vanishes along the boundary C where $r = c$.

Now, we have to determine $G(z)$ from the boundary condition (1.7c). In view of (3.2a) and (3.2b) the condition (1.7c) yields

$$\text{Re}\,[(1 - \lambda)\, zG'(z) + G(z) - \sum_{j=0}^{3} E_j z^j] = 0, \qquad \text{...(3.4)}$$

where

$$E_0 = (2 - \lambda)(1 + 6b^2 + 2b^4)/(2a^2) + \lambda\left(\frac{1}{3} + 4b^2 + 5b^4 + \frac{2}{3}b^6\right)\Big/c^2, \qquad \text{...(3.5}a\text{)}$$

$$E_1 = \lambda\, ab\,(1 + 5b^2 + 3b^4)/c^2 - 2b\,(1 + 2b^2)/(ac^2), \qquad \text{...(3.5}b\text{)}$$

$$E_2 = (2 - \lambda)\, b^2/c^4 - 2\lambda\, a^2b^2\,(1 + 2b^2)/c^6, \qquad \text{...(3.5}c\text{)}$$

$$E_3 = \lambda a^3b^3/(6c^8). \qquad \text{...(3.5}d\text{)}$$

Equation (3.4) can be satisfied by equating the expression between the square brackets to zero which leads to the linear differential equation

$$(1 - \lambda)\, zG'(z) + G(z) - \sum_{j=0}^{3} E_j z^j = 0. \qquad \text{...(3.6)}$$

Solving (3.6) we obtain

$$G(z) = m \sum_{j=0}^{3} \frac{E_j}{i + m}\, z^i, \qquad \text{...(3.7)}$$

where

$$m = \frac{1}{1 - \lambda}. \qquad \text{...(3.8)}$$

Having obtained the regular function $G(z)$, the complex potentials (3.1) and (3.2) are completely determined. It can be noticed that for a clamped boundary $\lambda = 1$ and the regular function is immediately obtained by putting $\lambda = 1$ in (3.6).

It is worthy to mention that when the loading over the concentric elliptic limacon is of the form

$$p_1 = p_0(x^2 - y^2) = p_0(z^2 + \bar{z}^2)/2, \qquad \text{...(3.9)}$$

the complex potentials can be dealt with on lines similar to those exemplified above. Thus having found the solutions of the two loadings (1.2a) and (3.9), it is possible by applying the principle of superposition, to derive the solution appropriate to the two parabolic loadings $p_1 = p_0 x^2$ and $p_1 = p_0 y^2$.

4. Central Values of Deflection, Moments and Shears

Letting z tend to zero in (3.7) and (3.1b), it can be easily shown that the deflection at the centre is given by

$$w_1(0) = 2\,\psi_1(0) = 36\,Ka^6\left[\frac{7}{9} + 7b^2 + 4b^4 - \frac{7}{9}b^6\right.$$
$$- \left(\frac{2}{3} + 8b^2 + 10b^4 + \frac{4}{3}b^6\right)\log\frac{a}{c} - (2-\lambda)\frac{c^2}{2a^2}(1 + 6b^2 + 2b^4)$$
$$\left. - \lambda\left(\frac{1}{3} + 4b^2 + 5b^4 + \frac{2}{3}b^6\right)\right]. \qquad \ldots(4.1)$$

Now, following Stevenson (1943), the bending moments and shearing forces at any point z of the plate may conveniently be written in the form

$$\overset{\frown}{r\theta} - \overset{\frown}{\theta r} = -\,4\,(1+\eta)\,P\;[2\mathrm{Re}\;\phi_j{}' + \partial^2 W_j/\partial z\partial\bar{z}], \qquad \ldots(4.2a)$$

$$\overset{\frown}{r\theta} + \overset{\frown}{\theta r} + 2i\overset{\frown}{\theta\theta} = -\,4\,(1-\eta)\,P\,[z\phi_j{}'' + z^2(\psi_j{}'' + W_j{}'')/r^2], \qquad \ldots(4.2b)$$

$$\overset{\smile}{rZ} - i\overset{\smile}{\theta Z} = -\,8Pz\,(\phi_j{}'' + \partial^3 W_j/\partial z^2\partial\bar{z})/r, \qquad \ldots(4.2c)$$

where $P = D/2h$, $2h$ is the thickness of the plate. Setting $z = 0$ in eqns. (4·2) yield

$$\overset{\frown}{xx}_0 = \overset{\frown}{yy}_0 = \overset{\smile}{xZ}_0 = \overset{\smile}{yZ}_0 = 0, \qquad \ldots(4.3a)$$

$$\overset{\frown}{xy}_0 - \overset{\frown}{yx}_0 = -\,8\,(1+\eta)\,P\phi_1{}'(0), \qquad \ldots(4.3b)$$

$$\overset{\frown}{xy}_0 + \overset{\frown}{yx}_0 = -\,4\,(1-\eta)\,P\,\psi_1{}''(0). \qquad \ldots(4.3c)$$

From (3.1) and (3.7) we obtain

$$\phi_1{}'(0) = 18Ka^4\left[\frac{1}{2}(2-\lambda)(1 + 6b^2 + 2b^4) + \frac{\lambda a^2}{c^2}\left(\frac{1}{3} + 4b^2 + 5b^4 + \frac{2}{3}b^6\right)\right.$$
$$\left. - (1 + 6b^2 + 2b^4)\log\frac{a}{c} - \frac{3}{4} - 7b^2 - \frac{17}{4}b^4\right], \qquad \ldots(4.4)$$

$$\psi_1{}''(0) = 36\,Ka^6\,\frac{b^2}{c^2}\left[\left(\frac{2\lambda}{3-2\lambda} - \frac{1}{3}\right)(1 + 2b^2)\,\frac{a^2}{c^2} - \frac{2-\lambda}{3-2\lambda}\right] \qquad \ldots(4.5)$$

Substituting from (4.4) and (4.5) in (4.3b) and (4.3c) the bending moments $\overset{\frown}{xy}_0$ and $\overset{\frown}{yx}_0$ are found.

5. Boundary Values of Slope, Moments and Shears

On the boundary of the plate, the conditions (1.6) with eqns. (4.2) imply that

$$\frac{\overset{\frown}{r\theta}}{\eta-\nu}=\frac{\overset{\frown}{\theta r}}{\eta\nu-1}=-\frac{P}{c}\frac{\partial W_2}{\partial r}=\frac{8P}{\nu-1}\ \mathrm{Re}\ [\phi_2'(z)], \qquad \text{...(5.1)}$$

$$\frac{\overset{\frown}{\theta\theta}}{1-\eta}=c\,\frac{\overset{\smile}{\theta Z}}{1-\nu}=-\frac{P}{c}\frac{\partial^2 w_2}{\partial r\,\partial\theta}, \qquad \text{...(5.2)}$$

$$\overset{\smile}{rZ}=-8\frac{P}{c}\mathrm{Re}\,[z\phi_2''(z)]\ , \qquad \text{...(5.3)}$$

where $z=ce^{i\theta}$. From (3.2*a*) and (3.7) we obtain

$$\mathrm{Re}\ [\phi_2'(z)]=18\,Ka^6\Big[E_0-\frac{1+6h^2+2b^4}{a^2}+2\left\{\frac{E_1}{2-\lambda}+\frac{b}{a^2c^2}(1+2b^2)\right\}c\cos\theta+\left(\frac{3E_2}{3-2\lambda}+2\frac{b^2}{c^4}\right)c^2\cos 2\theta+\frac{4E_3}{4-3\lambda}c^3\cos 3\theta\Big] \qquad \text{...(5.4)}$$

$$\mathrm{Re}\ [z\phi_2''(z)]=18\,Ka^6\Big[-\frac{1+6b^2+2b^4}{a^2}+2\left\{\frac{E_1}{2-\lambda}-\frac{b}{a^2c^2}(1+2b^2)\right\}c\cos\theta+2\left(\frac{3E_2}{3-2\lambda}-\frac{b^2}{c^4}\right)c^2\cos 2\theta+\frac{12E_3}{4-3\lambda}c^3\cos 3\theta\Big]. \qquad \text{...(5.5)}$$

Substituting from (5.4) in (5.1) and (5.5) in (5.3) we obtain the formulae of $r\theta$, $\overset{\frown}{\theta r}$ and $\overset{\smile}{rZ}$, at any point on the boundary.

From (5.1), (5.2) and (5.4) we get

$$\frac{\overset{\frown}{\theta\theta}}{1-\eta}=c\,\frac{\overset{\smile}{\theta Z}}{1-\nu}=-\frac{144\,KP}{\nu-1}a^6\Big[2\left\{\frac{E_1}{2-\lambda}+\frac{b}{a^2c^2}(1+2b^2)\right\}c\sin\theta+2\left(\frac{3E_2}{3-2\lambda}+\frac{2b^2}{c^4}\right)c^2\sin 2\theta+\frac{12E_3}{4-3\lambda}c^3\sin 3\theta\Big] \qquad \text{...(5.6)}$$

6. Limiting Case

Letting $b=0$ we obtain the particular case of a circular plate elastically restrained along its boundary according to the conditions (1.6) and subject to the same loading (1.2*a*) over a concentric circle. In this case the potential functions (3.1) and (3.2) reduce to

$$\phi_1(z) = 18Ka^4 z\left[\frac{1}{4} - \lambda\left(\frac{1}{2} - \frac{a}{3c^2}\right) - \log\frac{a}{c}\right] \qquad ...(6.1a)$$

$$\psi_1(z) = 18Ka^6\left[\frac{7}{9} - \frac{\lambda}{3} + \left(\frac{\lambda}{2} - 1\right)\frac{c^2}{a^2} - \frac{2}{3}\log\frac{a}{c}\right]; \qquad ...(6.1b)$$

$$\phi_2(z) = 18Ka^4 z\left[1 + \lambda\left(\frac{a^2}{3c^2} - \frac{1}{2}\right) - \log\frac{z}{c}\right] \qquad ...(6.2a)$$

$$\psi_2(z) = -18Ka^6\left[\frac{\lambda}{3} + \left(1 - \frac{\lambda}{2}\right)\frac{c^2}{a^2} + \frac{2}{3}\log\frac{z}{c}\right]. \qquad ...(6.2b)$$

The expressions given by (6.1) and (6.2) agree with those obtained by Bassali [1960; eqns. (7.1), (7.2), (7.3) and (7.4); p. 190].

REFERENCES

Bassali, W. A. (1959). Bending of a thin circular plate under hydrostatic pressure over a concentric ellipse. *Proc. Camb. phil. Soc.*, **55**, 110.

——— (1960). The transverse flexure of a thin circular plate subject to parabolic loading over a concentric ellipse. *J. appl. Math. Phys. (ZAMP)*, **11**, 176.

Bassali, W. A., and Barsoum, F. R. (1965). Bending of thin circular plate subject to quadratic loadings over a concentric ellipse. *Bull. Faculty Sci. Alex.*, 7, 1.

Bassali, W. A., and Dawoud, R. H. (1956*a*). Bending of a circular plate with an eccentric circular patch symmetrically loaded with respect to its centre. *Proc. Camb. phil. Soc.*, **52**, 584.

——— (1956). Thin circular plates under certain distributions of normal loading. *Mathematika*, **3**, 144.

Bassali, W. A., and Nassif, M. (1959). A thin circular plate normally and uniformly loaded over a concentric elliptic patch. *Proc. Camb. phil. Soc.*, **55**, 101.

Deverall, L. I. (1957). Solution of problems in bending of thin clamped plates by means of the method of Muskhelishvili. *J. appl. Mech.*, **24**; *Trans. ASME*, **79**, 295.

Ghose, S. (1925). On a problem of elastic circular plates. *Bull. Calcutta math. Soc.*, **16**, 63.

Gray, C. A. M. (1952). Polynomial approximations in elastic plates. *J. appl. Mech.*, **19**, 422.

Moskhelishvili, N. I. (1949). Some Basic Problems of the Mathematical Theory of Elasticity 3rd ed. (Moscow).

Shin-Min Jen (1949). Bending of circular plates. Ph.D. Thesis, University of Michigan, Mich. (U.S.A.).

Sokolnikoff, I. S. (1952). Some new methods of solution of two-dimensional problems in elasticity. *Bull. Am. math. Soc.*, **48**, 539.

Stevenson, A. C. (1943). The boundary couples in thin plates. *Phil. Mag.*, **34**, 105.

Tiffen, R. (1955). Some problems of thin clamped elastic plates. *Q. Jl. Mech. appl. Math.*, **8**, 237.

Yu, Y. Y. (1956). Flexural problems of thin plates under lateral loading solved by the complex variable method. *Proc. 9th Intern. Congr. appl. Mech., Brussels*, p. 378.