TRANSONIC FLOW FIELD FOR PARABOLIC SPINDLES BY METHOD OF CHARACTERISTICS

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The present work computes the solution of axisymmetric transonic flow with free-stream Mach number $M_\infty = 1$, past parabolic are spindles by the method of characteristics—the sonic line being computed by the parabolic method of Oswatitsch and Keune (1955). The pressure coefficient at the body has been compared with experimental results of Drougge and with other theoretical results.

1. INTRODUCTION

The linearization methods are often preferred in view of their simplicity for computing steady inviscid transonic flow past slender axisymmetric bodies, with sonic free-stream velocity. These methods originated from a work of Oswatitsch and Keune (1955) who simplified the basic small perturbation quasi-linear partial differential equations of mixed elliptic-hyperbolic type, under the physical assumption that the body experiences a constant acceleration. Various modifications and improvement of the method, like those of Spreiter and Alksne (1958) and Hosokawa (1964) followed. In a recent work Leelavati and Subramanian (1969) observed that the pressure distribution at the body for a family of parabolic spindles, computed by the linearization method of Hosokawa agreed better with experimental results, than that delivered by the local linearization method of Spreiter and Alksne. However, it was apparent from the computations of Leelavati and Subramanian that their results, although better than that of Spreiter and Alksne, still indicate much deviation from experimental results. In order to resolve this discrepancy, the present investigation has been undertaken, where the supersonic part of the flow field has been computed by the axisymmetric transonic method of characteristics of Oswatitsch (1956)—the sonic line being determined by the parabolic method of Oswatitsch and Keune (1955). The pressure coefficient at the body has been compared with experimental results and with other theoretical results, and characteristic network in the supersonic part of the flow field shown through figures. The true position of the shock appears to be situated a little upstream than that predicted by Hosokawa. It is to be noted that the only source of inaccuracy of the present characteristics method, which solves the nonlinear equations in the supersonic part of the plane to any desired degree of accuracy, lies in the approximate determination of the sonic line, which is known to be quite good for most practical purposes.

2. Formulation of the Problem

We consider transonic flow past a slender axisymmetric body at zero incidence with free-stream Mach number $M_\infty = 1$. Let us take the body axis as the axis of x and let y denote radial distance away from the body axis. Then the governing partial differential equations of a steady inviscid irrotational axisymmetric flow may be written as (Oswatitsch 1956*a*)

$$M_\infty = 1, \quad -\bar{y}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}}(v\bar{y}) = 0 \qquad \text{...(1)}$$

$$\frac{\partial \bar{u}}{\partial \bar{y}} - \frac{\partial \bar{v}}{\partial \bar{x}} = 0 \qquad \text{...(2)}$$

where

$$\left.\begin{aligned} \bar{u} &= \frac{1}{\pi\tau^2}\left(1 - \frac{U}{C^*}\right), \\ \bar{v}\bar{y} &= \frac{1}{\pi\tau^2}\,\frac{V}{C^*}\,y; \quad \bar{y} = y\sqrt{\pi\tau^2(\gamma+1)} \end{aligned}\right\} \qquad \text{...(3)}$$

Here $\bar{u}$ and $\bar{v}$ are reduced perturbation velocities, whose true values are U and V, and $\bar{y}$ denotes a reduced radial co-ordinate, and τ the thickness ratio.

The boundary condition at the body may be approximated as in the linear theory and written as

$$y = h(x), \qquad \bar{y} = \bar{h}(x),$$

$$M_\infty = 1, \quad \bar{v}\bar{y} = \frac{1}{\pi\tau^2}\cdot\frac{1}{2\pi}\cdot\frac{dF}{dx} \qquad \text{...(4)}$$

where $F(x)$ denotes cross-sectional area of the body.

The expression for the pressure coefficient is given by

$$C_p = -\frac{2u}{U_\infty} - \left(\frac{v}{U_\infty}\right)^2. \qquad \text{...(5)}$$

The expression for C_p given in (5) may be written in terms of the reduced quantities as

$$C_p = -\pi\tau^2\{2\bar{u} + (\gamma+1)(\pi\tau^2)^2 \,.\, \bar{v}^2\}. \qquad \text{...(6)}$$

3. Method of Characteristics

To compute the supersonic part of the flow field bounded by the body and the sonic line, we use the transonic method of characteristics developed by Oswatitsch (1956). The inclinations of the characteristics in the physical plane are given by

$$\frac{\xi_{\bar{y}}}{\xi_x} = -\sqrt{\bar{u}} \quad \text{and} \quad \frac{\eta_{\bar{y}}}{\eta_x} = \sqrt{\bar{u}} \qquad \text{...(7)}$$

respectively along the left and right running characteristics ξ = constant and η = constant, the suffixes denoting differentiation. The compatibility condition in the state plane is given by

$$d(\bar{v}\bar{y}) \mp \bar{y}d(\tfrac{2}{3}\bar{u}^{3/2}) = 0 \qquad \ldots(8)$$

the upper sign holding along the left-running and the lower sign along the right-running characteristics.

To begin the computation of the supersonic part of the flow field, the sonic line is determined according to the method of Oswatitsch and Keune (1955). We choose a point 1 on the sonic line lying very near the body [Fig. 1 (*a*)]. The value of $(\bar{v}\bar{y})$ at this point is taken approximately equal to that at the point of the body with the same abscissa, and plotted in the state plane [Fig. 1 (*b*)], *viz.*, in the

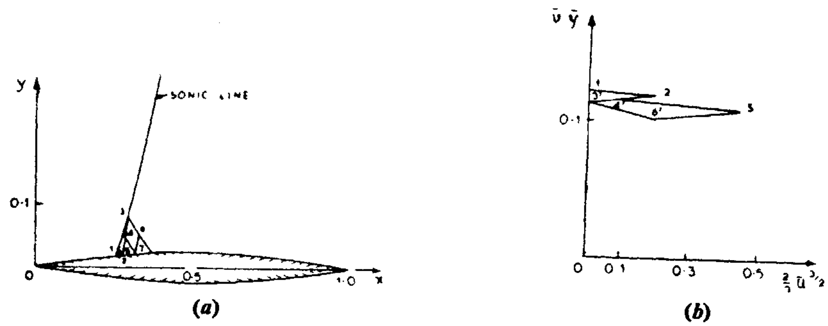


FIG. 1. Explanation of the method of characteristics. (*a*) Physical plane, (*b*) State plane.

$(\bar{v}\bar{y},\ \tfrac{2}{3}\bar{u}^{3/2})$ plane. The position of the point 2 at the body in the physical plane, is at first guessed and then improved by iteration. For this, we take the average of the ordinates of the points 1 and the guessed value of 2 and with this draw the right-running characteristics in the state plane, which is conveniently done with the help of an inclination diagram (Fig. 2). The value of $(\bar{v}\bar{y})$ at the point 2

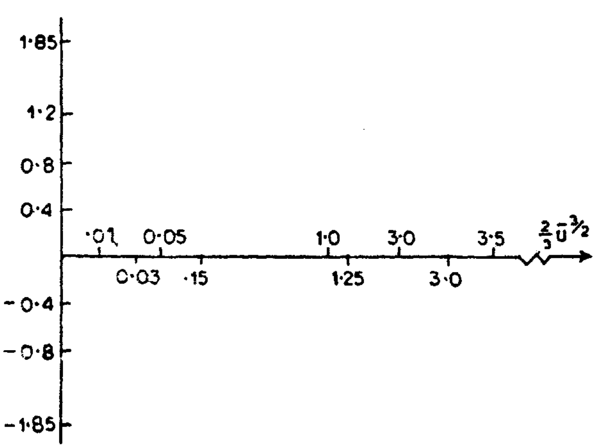


FIG. 2. Inclination diagram.

is known from the boundary condition at the body and consequently, the point $2'$ in the state plane is determined in the first approximation. Knowing $1'$, $2'$ which are the images of the points 1 and 2 in the state plane, the right-running characteristics through 1 in the physical plane is drawn with the average value of $(-1/\sqrt{\bar{u}})$ at the points $1'$ and $2'$ (use of an inclination diagram speeds up the drawing). With the new value of the ordinate 2, we again go over to the state plane and repeat the process until no change occurs in the position of the point 2. When the position of the point 2 is thus finally determined, we draw as the next step, the left-running characteristics through 2 to meet the sonic line at 3. As in the previous step, the position of the point 3 is guessed in the first approximation and with average of this ordinate of 3 and of 2 the left-running characteristic in the state plane through the point 2 is drawn. It is to be noted that on the sonic line $u = 0$, so that the image point $3'$ is determined in the state plane. Knowing $3'$ in the first approximation, the left-running characteristic through 2 is drawn in the physical plane. The whole process is repeated as before. In order that the network does not become too large, suitable interpolations have to be made and the point 4 and its image $4'$ are selected on the line 2–3 in both the planes. The entire network in Fig. 4 (*a*) is computed in this way. The boundary condition at the body is conveniently drawn on a graph paper in $(x, \bar{v}\bar{y})$–plane (Fig 3) which may be used during computation.

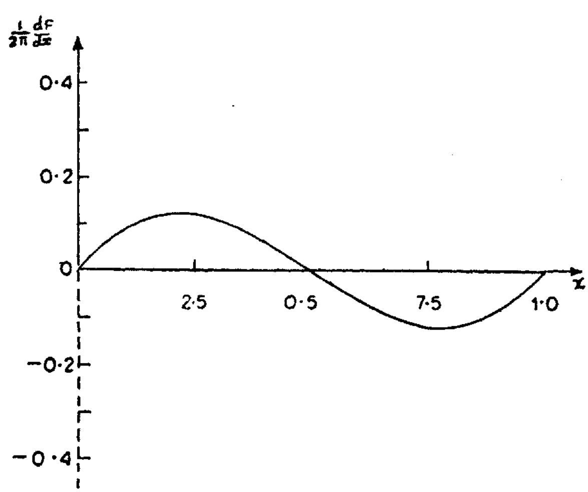


FIG. 3. Boundary condition.

4. DETERMINATION OF THE SONIC LINE

For axisymmetric transonic flow with $M_\infty = 1$, the gasdynamic equation for the perturbation potential ϕ may be written under the assumption of parabolic method (Oswatitsch and Keune 1955), *viz.*, $\phi_{xx} = a =$ constant > 0 as

$$\phi_{yy} + \frac{1}{y}\phi_y = a(\gamma + 1)\phi_x \qquad ...(9a)$$

along with the boundary condition

$$\lim_{y \to 0} y\phi_y(x, y) = \frac{1}{2\pi}\frac{dF}{dx}. \quad \ldots(9b)$$

The system (9) may be solved by superposition of elementary solutions, so as to yield (Zierep 1966)

$$\phi(x, y) = -\frac{1}{4\pi}\int_0^x \frac{F'(\xi)}{x-\xi}\exp\left(-a(\gamma+1)y^2/4(x-\xi)\right)d\xi. \quad \ldots(10)$$

After differentiating with respect to x and integrating by parts we obtain, noting that

$$F'(0) = 2\pi h(0)h'(0) = 0,$$

$$\phi(x, y) = -\frac{1}{4\pi}\int_0^x \frac{F''(\xi)}{x-\xi}\exp\left(-a(\gamma+1)y^2/4(x-\xi)\right)d\xi. \quad \ldots(11)$$

For parabolic arc spindles we consider body shape given by

$$h(x) = 2\tau(x - x^2), \quad 0 \leqslant x \leqslant 1, \quad \ldots(12)$$

$$= 0, \quad \text{otherwise},$$

for which the second derivative of the cross-sectional area is

$$F''(x) = 8\pi\tau^2(6x^2 - 6x + 1) \quad \ldots(13)$$

Substituting (13) in (11) we obtain on simplification

$$\phi(x, y) = \frac{1}{4\pi}\int_0^x \frac{F''(\xi) - F''(x)}{\xi - x}\exp\left(-a(\gamma+1)y^2/4(x-\xi)\right)d\xi.$$

$$-\frac{1}{4\pi}F''(x)\int_0^x \frac{\exp\left(-a(\gamma+1)y^2/4(x-\xi)\right)}{x-\xi}d\xi \quad \ldots(14)$$

The sonic line is then obtained on carrying out the integration and simplification as

$$[144x^2 + (12by^2 - 96)x]\exp(-by^2/4x)$$

$$= [16(6x^2 - 6x + 1) + 24by^2(2x - 1) + 3b^2y^4]\int_{by^2/4x}^{\infty}\frac{e^{-t}}{t}dt \quad \ldots(15)$$

where

$$b = a(\gamma + 1).$$

The values of Ei-function, *viz.*,

$$E_i(-z) = -\int_z^{\infty} \frac{e^{-t}}{t}\, dt$$

are taken from the tables (Jahnke *et al.* 1960).

5. Numerical Results

The computed flow fields for a parabolic arc spindle with $\tau = 0 \cdot 1$ and $\tau = 0 \cdot 07$ are shown in Figs. 4 (*a*), 4 (*b*), 6 (*a*) and 6 (*b*). The pressure coefficients at the body are shown in Figs. 5 and 7. It has been compared with the results of Leelavati and Subramanian (1969), Spreiter and Alksne (1958) and experimental results. From Figs. 5 and 7 it is found that the present result is in good agreement with experimental result, although some deviation is noted near the trailing edge. The computation of the characteristic network could not be continued after 80% of the chord, where the flow becomes subsonic. A shock-wave appears to be formed near this region and it appears that the position of the shock-wave is predicted moderately well by the method of Hosokawa, although the true position lies somewhat upstream.

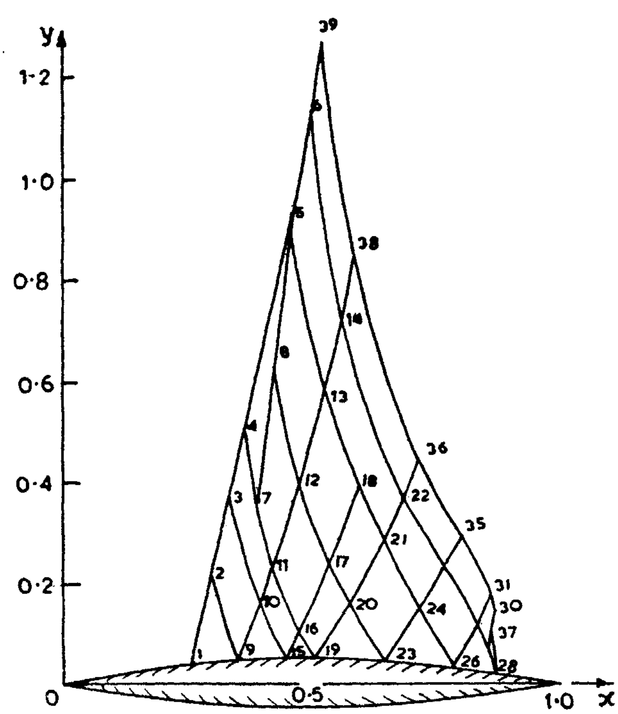
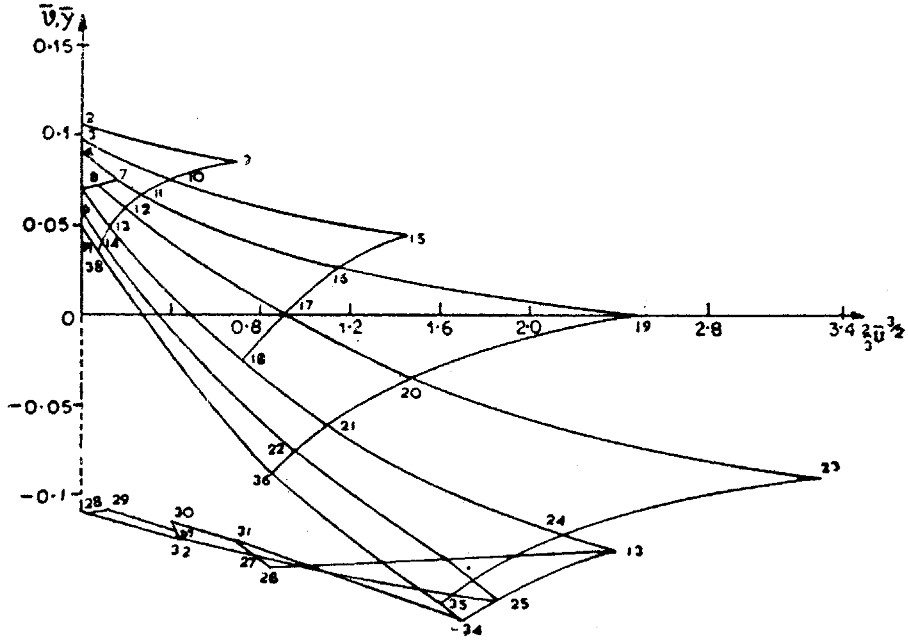


Fig. 4 (*a*). Characteristic network for parabolic spindle with $\tau = 0 \cdot 1$.



State plane

FIG. 4 (*b*). Characteristic network for parabolic spindle with $\tau = 0\cdot1$.

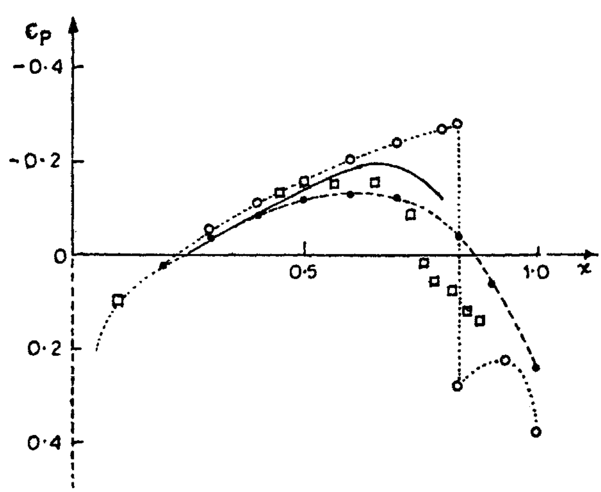
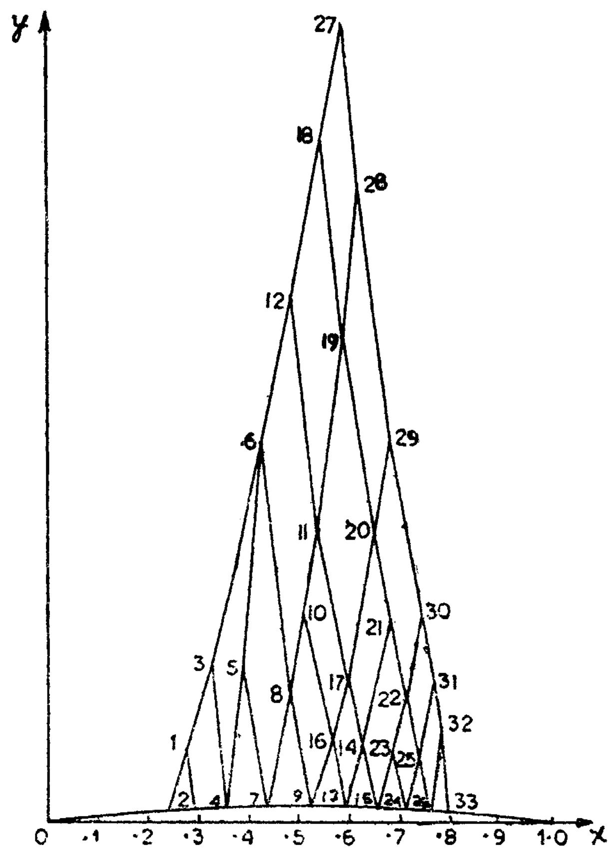
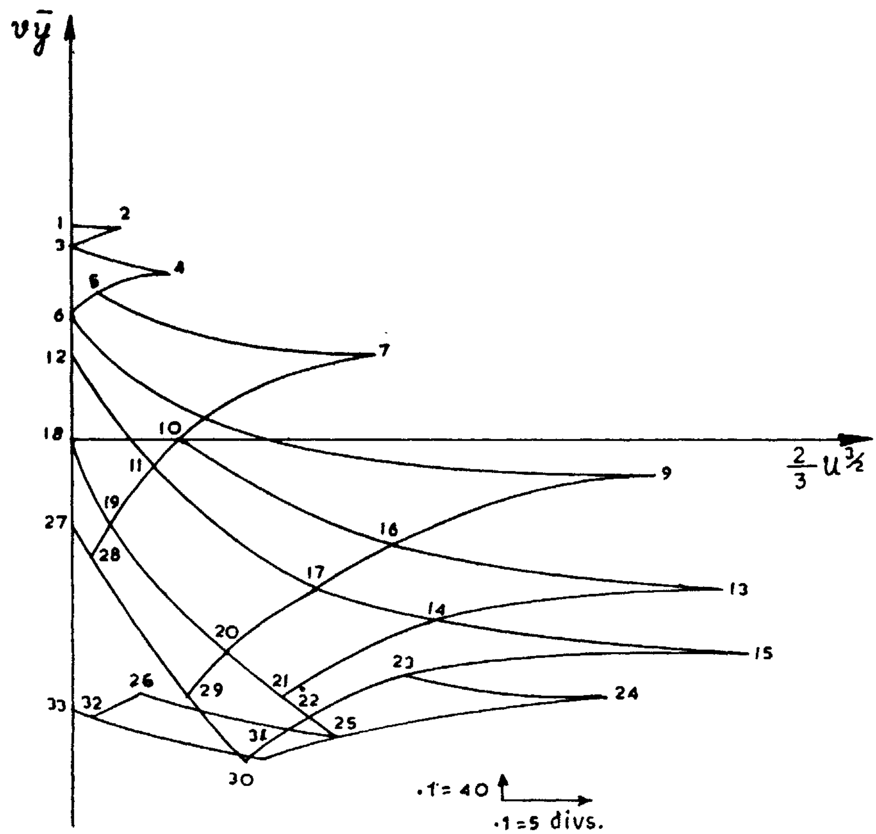


FIG. 5. Pressure distribution on a parabolic spindle for $\tau = 0\cdot1$.



Physical plane

FIG. 6 (*a*). Characteristic network for parabolic spindle for $\tau = 0\cdot07$.



State plane

FIG. 6 (*b*). Characteristic network for parabolic spindle for $\tau = 0 \cdot 07$.

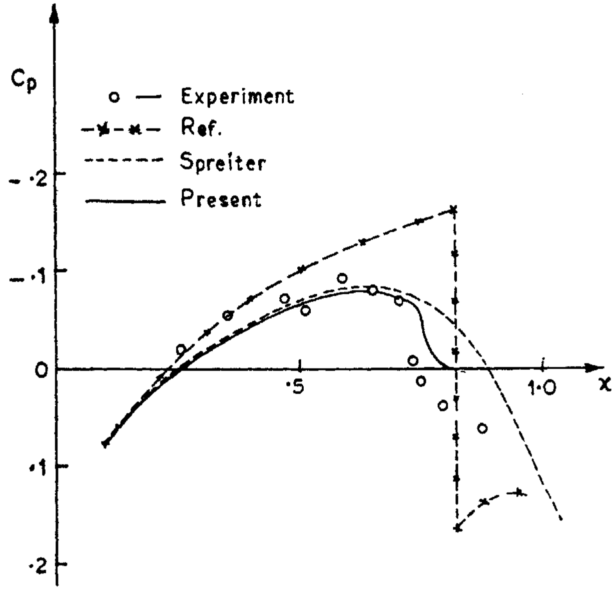


FIG. 7. Pressure distribution on a parabolic spindle for $\tau = 0{\cdot}07$.

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