A NEW SOLVABLE POTENTIAL FROM THE QUASILINEAR VAN DER POL EQUATION

by KAMAL S. RAJPUT*, *Department of Physics, Government Postgraduate College, Tikamgarh* 472001

(*Received* 16 *June* 1976)

Under certain conditions, the quasilinear van der Pol equation has been transformed into the radial *s*-wave Schrödinger equation and a new solvable potential for the latter is obtained. The *S*-matrix for the new potential is found to depend only upon the initial phase of the wave. A physical explanation of the wave-functional dependence of the new solvable potential is discussed on the basis of exterior and inner potentials.

1. INTRODUCTION

The transformation of a second-order differential equation, with its known solution, to a radial Schrödinger equation provides a potential-model applicable to the study of eigenvalue problem for the bound states and the evaluation of scattering parameters in a collision theory. The choice of a potential-model, however, depends upon the best fit for the experimental results.

Bhattacharjie and Sudarshan (1962) devised a technique for the transformation of a linear second-order differential equation to the radial Schrödinger equation. Another technique for the same linear type of differential equations was developed by Bose (1963). Both these techniques have been used extensively (Sharma and Varma 1970; Sharma 1970*a*, *b*; Khan 1973) to obtain solvable potentials for the Schrödinger equation and it has been invariably proved that both techniques give the same set of potentials.

So far as known to the author, no effort has been made to transform the non-linear differential equations. As the Schrödinger equation is linear, the difficulty appears to be that of linearization. As a preliminary effort in this direction, the transformation of the quasilinear van der Pol equation is attempted in this paper. The transformation yields a new solvable potential. The *S*-matrix for the new potential is derived and a physical explanation for the interesting nature of the new solvable potential is discussed.

2. THE QUASILINEAR VAN DER POL EQUATION

The general form of a time-dependent quasilinear differential equation is given as (Pipes 1958)

$$\frac{d^2u}{dt^2} + \mu f\left(u, \frac{du}{dt}\right) + \omega^2 u = 0 \qquad \ldots(1)$$

* Present address: Department of Physics, Maharaja College, Chhatarpur 471001.

where the parameter μ is a small constant and

$$f\left(u, \frac{du}{dt}\right)$$

is a given function of u and its time-derivative.

The quasilinear van der Pol equation, which is a particular form of (1), is given as (Pipes 1958)

$$\frac{d^2u}{dt^2} - \mu(1 - u^2)\frac{du}{dt} + u = 0 \qquad ...(2)$$

with the solution

$$u(t) = \frac{a_0 e^{t\mu/2} \sin(t + \phi_0)}{\{1 + (a_0^2/4)(e^{\mu t} - 1)\}^{1/2}}. \qquad ...(3)$$

The van der Pol equation (2), being time-dependent, has to be reduced to a convenient radial form. To effect this reduction, let us imagine that the quasilinear vibrations represented by (2) are executed by a mechanical vibrator just touching the surface of a liquid at a point called the origin. As the vibrator vibrates, a disturbance in the form of waves propagates from the origin in all directions in the two-dimensional plane of the surface. Let the disturbance propagate with a fixed velocity v characteristic of the liquid surface. Following the arguments of Pipes (1958, pp. 61–63) on the propagation of periodic disturbances in the form of waves we have the wave function

$$u(r, t) = A \exp\left\{i\omega\left(t - \frac{r}{v}\right)\right\}. \qquad ...(4)$$

The wave propagates in a way to give

$$t - \frac{r}{v} = \text{const.} \qquad ...(5)$$

and a phase velocity given by

$$\frac{dr}{dt} = v. \qquad ...(6)$$

We have from (5) and (6)

$$\frac{\partial u}{\partial t} = v\frac{\partial u}{\partial r} \qquad ...(7)$$

and the equation of propagation of waves

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial r^2}. \qquad ...(8)$$

Substituting (7) and (8) in (2) and replacing the partial derivatives by perfect differentials, we get

$$\frac{d^2u(r)}{dr^2} - \frac{\mu}{v}\{1 - u^2(r)\}\frac{du(r)}{dr} + \frac{u(r)}{v^2} = 0 \qquad ...(9)$$

If we have the initial condition $r = 0$ at $t = 0$ to give $t = r/v$. The solution of (9) can be obtained from (3) as

$$u(r) = \frac{a_0 \exp(\mu r/2v) \sin(r/v + \phi_0)}{[1 + (a_0^2/4)\{\exp(\mu r/v) - 1\}]^{1/2}} \qquad \text{...(10)}$$

Equation (9) represents the desired radial form of the van der Pol equation with a solution given by (10).

3. Transformation and Solvable Potential

We consider the radial van der Pol equation (9) with the solution (10) under the condition

$$\frac{du(r)}{dr} = z(r) \cdot u(r) \qquad \text{...(11)}$$

Using the method of elimination by differentiation (Protter and Morrey 1964, p. 699), eqn. (9) transforms to

$$\frac{d^2u(r)}{dr^2} + \left[\frac{1}{v^2} - \frac{\mu}{v}(1 - u^2(r)\ Z(r)\right] u(r) = 0. \qquad \text{...(12)}$$

Equation (12) can be compared with the *s*-wave Schrödinger equation

$$\frac{d^2R(r)}{dr^2} + [K^2 - V(r)]\ R(r) = 0 \qquad \text{...(13)}$$

[with units such that energy $E = k^2$]

where

$$k^2 = \frac{1}{v^2} \qquad \text{...(14)}$$

and

$$V(r) = \frac{\mu}{v}(1 - u^2(r))\, Z(r). \qquad \text{...(15)}$$

From (11) and (15) we get

$$V(r) = \frac{\mu}{v}\ (1 - u^2(r)) \cdot \frac{1}{u(r)}\ \frac{du(r)}{dr}. \qquad \text{...(16)}$$

Eliminating the function $u(r)$ by using the solution (10) in (16)

$$V(r) = \frac{\mu}{v^2}\left[1 - \frac{a_0^2 \exp(\mu r/v) \sin^2(r/v - \phi_0)}{1 + (a_0^2/4)\{\exp(\mu r/v) - 1\}}\right]$$
$$\times \left[\frac{\mu}{2} + \cot(r/v + \phi_0) - \frac{\mu a_0^2}{8}\right.$$
$$\left.\times \frac{\exp(\mu r/v)}{1 + (a_0^2/4)\{\exp(\mu r/v) - 1\}}\right]. \qquad \text{...(17)}$$

It is of interest to note that the radial van der Pol equation (9) with the solution (10) has been directly converted to the radial s-wave Schrödinger equation (13), under the condition (11), without resorting to functional transformations undertaken in the Bhattacharjie and Sudarshan's as well as in the Bose's techniques. The function (1) remains the common solution of the radial van der Pol equation (9) and the Schrödinger equation (13). This leads to construct potential functions for which the Schrödinger wave equation represents waves similar to those represented by a radial van der Pol equation. This further requires an analogy between the classical and quantum wave fields (Schiff 1955).

4. Solution and S-Matrix

The solution (10) of the radial van der Pol equation (9) can be written as

$$u(r) = \frac{a_0 \left[e^{i(r/v+\phi_0)} - e^{-i(r/v+\phi_0)}\right]}{2i\left[e^{-\mu/rv} - \frac{a_0^2}{4}(1-e^{-\mu r/v})\right]^{1/2}} \qquad ...(18)$$

The a symptotic form of the solution (18) is

$$u(r) \underset{r\to\infty}{=} - e^{i(r/v+\phi_0)} + e^{-i(r/v+\phi_0)}. \qquad ...(19)$$

Writing $k = \frac{1}{v}$ from (14) we get

$$u(r) \underset{r\to\infty}{=} - e^{i\phi_0}\, e^{ikr} + e^{-i\phi_0}\, e^{-ikr} \qquad ...(20)$$

which is of the form

$$u(r) \underset{r\to\infty}{=} Pe^{ikr} + Qe^{-ikr} \qquad ...(21)$$

and the S-matrix is given by

$$S(k) = -\frac{Q}{P} = e^{-2i\phi_0}. \qquad ...(22)$$

It is seen that the S-matrix is independent of k and depends only upon the initial phase ϕ_0.

5. Discussion

Heber and Weber (1959) have considered the potential function V as consisting of an exterior potential V_e and an inner potential V_i, such that

$$V = V_e + V_i \qquad ...(23)$$

where, as the meaning of the exterior potential is obvious, the inner potential V_i expresses the interaction arising from the distribution of matter itself. Thus, V_i depends upon the wave function u, or rather upon u^*u, according to the Poisson equation. In the general treatment of the Schrödinger equation V_i is neglected and V is identified with V_e. Strictly speaking, V cannot be identified with V_e and it is necessary to find the simultaneous solution for the system consisting of the Poission equation

$$\Delta^2 V_i = u^* u \qquad \ldots(24)$$

and the Schrödinger equation (13) with V given by (23).

In this paper, the transformation of the van der Pol equation leads to equation (12) with a solvable potential (15) in the form

$$V(r) = \frac{\mu}{v} Z(r) - \frac{\mu}{v} Z(r) u^2(r). \qquad \ldots(25)$$

Assuming the potential (25) to be of the form (23) we can write

$$V_e = \frac{\mu}{v} Z(r) \qquad \ldots(26)$$

and

$$V_i = -\frac{\mu}{v} Z(r) u^2(r). \qquad \ldots(27)$$

As anticipated, the form of the inner potential given by (27), for real values of $u(r)$, indicates its dependence upon the distribution of matter itself.

The foregoing interpretation leads to an exterior potential (26), written explicitly from (11) and (10) as

$$V_e = \frac{\mu}{v^2}\left[\frac{\mu}{2} + \cot\left(\frac{r}{v} + \phi_0\right) - \frac{a_0 \mu}{8} \cdot \frac{\exp(\mu r/v)}{1 + (a_0^2/4)[\exp(\mu r/v) - 1]}\right]. \qquad \ldots(28)$$

As it is evident, a detailed study is warranted to probe deeper into the relationship of classical wave equations, particularly the quasilinear and non-linear wave equations with the quantum wave equations.

Acknowledgement

The author is thankful to Prof. R. C. Varma for stimulating discussions and encouragement.

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