CONSTRAINED EXTREMAL PROBLEMS FOR FUNCTIONS WITH POSITIVE REAL PART

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Let P be the class of regular analytic functions in the unit disc $E = \{z \mid |z| < 1\}$ which take the value 1 at the origin with real part positive in E and let P_1 and P_ξ denote subclasses of P which, respectively, have fixed value of derivative at the origin and fixed value at a fixed point $\zeta \in E$. In this paper variational formulae for the classes P_1 and P_ξ have been obtained and complete solutions for the extreme values of $\text{Re}\, F(p(z))$, $p \in P_1$ or $p \in P_\xi$, have been given when $F(w)$ is an analytic function in $\text{Re}\, w > 0$ and $z \in E$.

INTRODUCTION

In the unit disc $E = \{z \mid |z| < 1\}$ let P denote the class of regular analytic functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ with $\text{Re}\, p(z) > 0$ and let P_1 and P_ξ denote subclasses of P which, respectively, have fixed value of p_1 which may be taken to be real and take a given value w at a fixed point $\zeta \in E$. Robertson (1962) and Sakaguchi (1964) have obtained variational formulae for the class P and proved the following.

Theorem A—Let $F(u)$ be analytic in $\text{Re}\, u > 0$. Then

$$\begin{matrix}\min \\ \max\end{matrix}_{\substack{p \in P \\ |z| = r < 1}} \text{Re}\, F(p(z)) = \begin{matrix}\min \\ \max\end{matrix}_{|z| = r} \text{Re}\, F\left(\frac{1+z}{1-z}\right) \qquad \ldots(1)$$

Theorem B—Let $F(u, v)$ be analytic with respect to u, v in $\text{Re}\, u > 0$, $|v| < \infty$, then

$$\begin{matrix}\min \\ \max\end{matrix}_{\substack{p \in P \\ |z| = r < 1,}} \text{Re}\, F(p(z),\, zp'(z)) = \begin{matrix}\min \\ \max\end{matrix}_{\alpha,\, \theta,\, \phi.} \text{Re}\, F\{P_0(z),\, zP_0'(z)\} \qquad \ldots(2)$$

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where

$$P_0(z) = \frac{1+\alpha}{2}\left(\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right) + \frac{1-\alpha}{2}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right) \qquad \ldots(3)$$

$$z = re^{i\phi},\ -1 \leqslant \alpha \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant 2\pi.$$

For the sake of completeness we give below the variational formula of Sakaguchi:

Theorem C—Let $p(z) \in P$ then there exists a function $p^*(z) = p(z) + \delta p(z)$ belonging to P and with $\delta p(z)$ of the form

$$\frac{2}{\rho}\,\delta p(z) = \varepsilon\left[p(z)\frac{1+\bar{\beta}z}{1-\bar{\beta}z} - \overline{p(\beta)}\left(p(z) - \frac{1+\bar{\beta}z}{1-\bar{\beta}z}\right) - 1\right]$$
$$-\bar{\varepsilon}\left[p(z)\frac{\beta+z}{\beta-z} + p(\beta)\left(p(z) - \frac{\beta+z}{\beta-z}\right) - 1\right] + 0(1) \quad \ldots(4)$$

where β and $\in$ are arbitrary complex numbers such that $|\beta| < 1$, $|\varepsilon| = 1$ and ρ is a sufficiently small positive number.

It is well known that if one has to solve extremal problems for the classes P_1 and P_ξ with the help of the variational formulae of Robertson (1962) or Sakaguchi (1964), then one has to use the method of undetermined multipliers by introducing an unknown parameter. The introduction of such unknown parameter usually leads to complications and complete solution is not easily obtained. In the present paper we show that a function either of the class P_1 or P_ξ can be represented in terms of an arbitrary function of the class P. This enables us to obtain the variational formulae for the functions belonging either to the class P_1 or P_ξ. Using these variational formulae we give complete solution for the problems analogous to Theorem A for the class P_1 and P_ξ. It may be remarked that for $\xi = 0$, P_ξ reduces to the class P_1 and hence we shall confine ourselves to the class P_ξ and distinguish the case when $\xi = 0$. Further, for $p \in P$ and $|p'(0)|$ fixed we give sharp upper bound on $|zp'(z)|/\text{Re}\, p(z)$ which reduces to known result (Robertson 1963) when $|p'(0)|$ is arbitrary.

Lemma 1—Let $p(z) \in P$ be any arbitrary function, then

$$f(z) = \frac{(1+a_1)(1-z) + (1-a_1)(1+z)p(z)}{(1+a_1)(1+z) + (1-a_1)(1-z)p(z)},\quad -1 \leqslant a_1 \leqslant 1,\ z \in E, \qquad \ldots(5)$$

belongs to P_1 for $p_1 = -2a_1$ and

$$f(z) = \frac{[f(\xi)(1-\bar{\xi}z)(1-\bar{F}) - \overline{f(\xi)}(z-\xi)(1-F)] + p(z)[f(\xi)(1-\bar{\xi}z)(1+\bar{F}) + \overline{f(\xi)}(z-\xi)(1+F)]}{[(1-\bar{\xi}z)(1-\bar{F}) + (z-\xi)(1-F)] + p(z)[(1-\bar{\xi}z) \times (1+\bar{F}) - (z-\xi)(1+F)]}, \quad \ldots(6)$$

where

$$F = -\frac{1}{\xi}\,\frac{1-f(\xi)}{1+f(\xi)}, \qquad \ldots(7)$$

belongs to P_ξ.

PROOF: If we put $\xi = 0$ in (6) and use the fact that $f(z) \in P_1$ with $p_1 = -2a_1$, we obtain (5). Hence we shall only prove (6). Let $f \in P_\xi$ and $\xi \in E$, then

$$F(z, \xi) = \frac{f(z) - f(\xi)}{f(z) + f(\xi)} \frac{1 - z\bar{\xi}}{z - \xi}, \qquad \ldots(8)$$

is regular in E and $|F(z, \xi)| < 1$. Since

$$F(0, \xi) = -\frac{1}{\xi} \frac{1 - f(\xi)}{1 + f(\xi)} = F, \qquad \ldots(9)$$

is different from zero in general, the function

$$\phi(z) = \frac{F(z, \xi) - F(0, \xi)}{1 - \overline{F(0, \xi)}\, F(z, \xi)}, \qquad \ldots(10)$$

is regular in E and satisfies the conditions $\phi(0) = 0$, $|\phi(z)| < 1$.

Hence there exists a function $p(z) \in P$ such that

$$\phi(z) = \frac{p(z) - 1}{p(z) + 1}. \qquad \ldots(11)$$

If in (10) we replace $\phi(z)$ from (11) and $F(z, \xi)$ from (8) and solve for $f(z)$ we obtain (6).

As $p(z)$ in (5) and (6) is a function of the class P for which variational formula has been given by Sakaguchi (1964) we obtain the following:

Theorem 1—Let $f(z) \in P_1$ with $p_1 = -2a_1$, then there exists a function $f^*(z) = f(z) + \delta f(z)$, $f^*(z) \in P_1$, with $\delta f(z)$ given by

$$\begin{aligned}\frac{2}{\rho}\delta f(z) = \varepsilon X(z, \bar{\beta})\Big[&\Big(1 - a_1 \frac{\bar{\beta} + z}{1 + \bar{\beta}z}\Big)(\phi(-a_1, \bar{\beta}, z) f(z) - 1) \\ &- \Big(1 + a_1 \frac{\bar{\beta} + z}{1 + \bar{\beta}z}\Big)(f(z) - \phi(a_1, \bar{\beta}, z))\overline{f(\beta)}\Big] \\ - \bar{\varepsilon}\, Y(z, \beta)\Big[&\Big(1 - a_1 \frac{1 + \beta z}{\beta + z}\Big)(\psi(-a_1, \beta, z) f(z) - 1) \\ &+ \Big(1 + a_1 \frac{1 + \beta z}{\beta + z}\Big)(f(z) - \psi(a_1, \beta, z)) f(\beta)\Big] + 0(1) \quad \ldots(12)\end{aligned}$$

where

$$X(z, \beta) = \frac{\{(1 + z) - (1 - z) f(z)\}(1 + \bar{\beta}z)}{z(1 - a_1^2)\{(1 + \bar{\beta}) - (1 - \bar{\beta})\overline{f(\beta)}\}},$$

$$Y(z, \beta) = \frac{\{(1 + z) - (1 - z) f(z)\}(\beta + z)}{z(1 - a_1^2)\{(1 + \beta) - (1 - \beta) f(\beta)\}},$$

$$\phi(a_1, \bar{\beta}, z) = \frac{(1 - a_1 z) + a_1(1 - 2a_1 z + z^2)\bar{\beta} + z(z - a_1)\bar{\beta}^2}{(1 - \bar{\beta}z)\{(1 + a_1 z) + (a_1 + z)\bar{\beta}\}},$$

$$= \psi\Big(a, \frac{1}{\bar{\beta}}, z\Big).$$

β and ε are arbitrary complex numbers such that $\beta \in E$, $|\varepsilon| = 1$ and $\rho > 0$ is sufficiently small.

Similarly using (6) we obtain the following:

Theorem 2—Let $f(z) \in P_\xi$, then there exists a function $f^*(z) = f(z) + \delta f(z)$, $f^*(z) \in P_\xi$, with $\delta f(z)$ given by

$$\frac{2}{\rho}\,\delta f(z) = X(z,\bar\beta)\,\varepsilon\,[\phi_1(z,\bar\beta) f(z) - X_1(z,\bar\beta) - \overline{f(\beta)}\,\{X_2(z,\bar\beta) f(z) - \phi_2(z,\beta)\}] - Y(z,\beta)\,\bar\varepsilon\,[\psi_1(z,\beta) f(z) - Y_1(z,\beta) + f(\beta)\,\{Y_2(z,\beta) f(z) - \psi_2(z,\beta)\}] + o(1) \qquad \ldots(13)$$

where

$$X(z,\beta) = \frac{\{(A - z\bar A) f(z) - (B + z\bar B)\}\,(1 + \bar\beta z)}{(1 - \bar\xi z)(z - \xi)(1 - |F|^2)\,\mathrm{Re} f(\xi)\,\{(\bar A - \bar\beta A)\overline{f(\beta)} - (\bar B + \bar\beta B)\}},$$

$$Y(z,\beta) = \frac{\{(A - z\bar A) f(z) - (B + z\bar B)\}\,(\beta + z)}{(1 - \bar\xi z)(z - \xi)(1 - |F|^2)\,\mathrm{Re} f(\xi)\,\{(A - \beta\bar A) f(\beta) - (B + \beta\bar B)\}},$$

and

$$A = 1 + \bar F + \xi(1 + F),$$

$$B = f(\xi)(1 + \bar F) - \xi\,\overline{f(\xi)}\,(1 + F)$$

$$X_1(z,\bar\beta) = |f(\xi) - \xi F\,\overline{f(\xi)}|^2 + \frac{[f(\xi) - \xi F\overline{f(\xi)}][f(\xi)\bar F - \xi\overline{f(\xi)}]\,\bar\beta + z\,[\overline{f(\xi)} - \bar\xi\bar F f(\xi)] \times [\overline{f(\xi)}F - \bar\xi f(\xi)]}{(1 + \bar\beta z)}$$

$$= Y_1\left(z, \frac{1}{\bar\beta}\right)$$

$$X_2(z,\bar\beta) = |1 + \xi F|^2 - \frac{(1+\xi F)(\bar F + \xi)\,\bar\beta + (1 + \bar\xi\bar F)(F + \bar\xi)\,z}{(1 + \bar\beta z)} = Y_2\left(z, \frac{1}{\bar\beta}\right)$$

$$\phi_1(z,\bar\beta) = \frac{[(1 + \xi F) - (\bar\xi + F)z]\,[(\overline{f(\xi)} - \bar\xi\bar F f(\xi)) + (f(\xi)\bar F - \xi\overline{f(\xi)})\,\bar\beta] + \bar\beta z\,[(1 + \bar\xi\bar F)z - (\xi + \bar F]\,[(\overline{f(\xi)}F - \bar\xi f(\xi)) + (f(\xi) - \xi F\overline{f(\xi)})\,\bar\beta]}{(1 - \bar\beta^2 z^2)}$$

$$= \psi_1\left(z, \frac{1}{\bar\beta}\right) = -\,\phi_2\left(\frac{1}{\bar\beta}, \frac{1}{z}\right) = -\,\psi_2\left(\frac{1}{\bar\beta}, z\right)$$

β and ε are arbitrary complex numbers such that $\beta \in E$, $|\varepsilon| = 1$ and ρ is sufficiently small positive number.

However, it is rather easier to handle functions of the class P than the functions belonging either to the class P_1 or P_ξ, hence for extremal problems pertaining either to P_1 or P_ξ, we use Lemma 1 and the variational formula for $p(z)$ as given by Sakaguchi (1964).

We shall now prove the following:

Theorem 3—Let $c \in E$, $c \neq \xi$ be fixed and let $F(w)$ be an analytic function of w in Re $w > 0$. If $f(z) \in P_\xi$ gives maximum or minimum Re $F(f(c))$ and $\lambda = F_w(f(c)) \neq 0$, then $f(z)$ is of the form

$$f(z) = \frac{[f(\xi)(1-\bar{\xi}z) + (z-\xi)\overline{f(\xi)}\,F] - [(1-\bar{\xi}z)f(\xi)\bar{F} + (z-\xi)\overline{f(\xi)}]\,\eta z}{[(1-\bar{\xi}z) - (z-\xi)F] + [(z-\xi) - (1-\bar{\xi}z)\bar{F}]\,\eta z} \qquad \ldots(14)$$

with $|\eta| = 1$. If $f(z)$ gives the minimum then

$$\eta = -\frac{\bar{c}}{c}\,\frac{\phi(c,\xi,F)\,|\lambda_1| - 2i\,\mathrm{Im}\,[c\lambda_1\{(1-\xi\bar{c}) - (\bar{c}-\bar{\xi})\bar{F}\}\{(\bar{c}-\bar{\xi}) - (1-\xi\bar{c})F\}]}{\lambda_1[(1-\xi\bar{c}) - (\bar{c}-\bar{\xi})\bar{F}]^2 - \bar{\lambda}_1[(c-\xi) - (1-\bar{\xi}c)\bar{F}]^2\,\bar{c}^2} \qquad \ldots(15)$$

and if it gives the maximum, then

$$\eta = \frac{\bar{c}}{c}\,\frac{\phi(c,\xi,F)\,|\lambda_1| + 2i\,\mathrm{Im}\,[c\lambda_1\{(1-\xi\bar{c}) - (\bar{c}-\bar{\xi})\bar{F}\}\{(\bar{c}-\bar{\xi}) - (1-\xi\bar{c})F\}]}{\lambda_1[(1-\xi\bar{c}) - (\bar{c}-\bar{\xi})\bar{F}]^2 - \bar{\lambda}_1[(c-\xi) - (1-\bar{\xi}c)\bar{F}]^2\,\bar{c}^2} \qquad \ldots(16)$$

where

$$c\lambda_1 = \lambda(1-\bar{\xi}c)(c-\xi)$$

and

$$\phi(c,\xi,F) = (1-|c|^2)\,[(1+|c|^2 - 2\mathrm{Re}\,\bar{\xi}c) + |F|^2\,\{|\xi|^2(1+|c|^2) - 2\mathrm{Re}\,\bar{\xi}c\} - 2\mathrm{Re}\,\{(1-\xi\bar{c})(c-\xi)F\}].$$

Proof: We note that compactness of the class P_ξ ensures the existence of the extremal functions. Further, if δ denotes the variation, we have

$$\delta\,\mathrm{Re}\,F(f(c)) = \mathrm{Re}\,\{\lambda\,\delta f(c)\} + o(1),\ \lambda = F_w(f(c)) \qquad \ldots(17)$$

and from (6)

$$\delta f(c) = \frac{[4(1-\bar{\xi}c)(c-\xi)(1-|F|^2)\,\mathrm{Re}\,f(\xi)]\,\delta p(c)}{\{(1-\bar{\xi}c)(1+\bar{F}) - (c-\xi)(1+F)]\,p(c) + [(1-\bar{\xi}c)(1-\bar{F}) + (c-\xi)(1-F)]\}^2} + o(1). \qquad \ldots(18)$$

Using, the expression for $\delta p(c)$ from Theorem C we get

$$\delta \operatorname{Re} F(f(c))$$

$$= \frac{\rho}{2} \operatorname{Re} \bar{\epsilon}\Big[\Big\{\bar{\lambda}\bar{X}\Big(\overline{p(c)}\,\frac{1+\bar{c}\beta}{1-\bar{c}\beta}-1\Big)-\lambda X\Big(p(c)\,\frac{\beta+c}{\beta-c}-1\Big)\Big\}$$

$$-\,p(\beta)\Big\{\bar{\lambda}\bar{X}\Big(\overline{p(c)}-\frac{1+\bar{c}\beta}{1-\bar{c}\beta}\Big)+\lambda X\Big(p(c)-\frac{\beta+c}{\beta-c}\Big)\Big\}\Big]$$

$$+\,o(1), \qquad \ldots(19)$$

where

$$X=\frac{4(1-c\bar{\xi})(c-\xi)(1-|F|^2)\operatorname{Re} f(\xi)}{[\{(1-\bar{\xi}c)(1+\bar{F})-(c-\xi)(1+F)\}p(c)+\{(1-\bar{\xi}c)(1-\bar{F})+(c-\xi)(1-F)\}]^2} \qquad \ldots(20)$$

$|\varepsilon|=1$ $\beta\in E$ and $\rho>o$ is arbitrarily small.

The usual variational argument then gives that the expression within the square brackets in the right-hand side of (19) must vanish. Since in this expression $\beta\in E$ is arbitrary, replacing it by z we obtain

$$p(z)=\frac{\bar{\lambda}\bar{X}\Big(\overline{p(c)}\,\frac{1+cz}{1-\bar{c}z}-1\Big)-\lambda X\Big(p(c)\,\frac{z+c}{z-c}-1\Big)}{\bar{\lambda}\bar{X}\Big(\overline{p(c)}-\frac{1+\bar{c}z}{1-\bar{c}z}\Big)+\lambda X\Big(p(c)-\frac{z+c}{z-c}\Big)}=\frac{A(z)}{B(z)}. \qquad \ldots(21)$$

It is readily verified that for $|z|=1$, $A(z)$ is pure imaginary and $B(z)$ is real. As $p(z)$ is a regular analytic function in E with $\operatorname{Re} p(z)>0$ which, as given by (21), is a rational function of z, it must have at least one pole on $|z|=1$ and all its poles on $|z|=1$ must be simple. Since $A(z)$ has no poles on $|z|=1$, $B(z)$ must have at least one zero on $|z|=1$. Further, because $B(z)$ is quadratic in z, it can have atmost two zeros. We proceed to show that $B(z)$ has a zero of order two on $|z|=1$.

Lemma 2—If $f(z)$ gives maximum $\operatorname{Re} F(f(c))$ then $B(z)\geqslant 0$ on $|z|=1$ and if $f(z)$ gives minimum $\operatorname{Re} F(f(c))$, then $B(z)\leqslant 0$ on $|z|=1$.

PROOF: We shall prove the case of the maximum for the other case is exactly similar and can easily be derived. We note that the extremal function $f(z)\in P_\xi$ will be of the form given by (6) where $p(z)\in P$ is some function. Let $p(z)\in P$ in (6) correspond to the extremal function $f(z)$. Then

$$p^*(z)=\frac{p(z)+\rho\,\dfrac{1+\bar{\alpha}z}{1-\bar{\alpha}z}}{1+\rho},\ \rho>0,\ \alpha\in E, \qquad \ldots(22)$$

also belongs to P and we note by $f^*(z)$ the function $f(z)$ given by (6) for which $p(z)$ is replaced by $p^*(z)$ given by (22). It is easily verified that

$$\delta f(c) = f^*(c) - f(c)$$

$$= \frac{4(1-\bar{\xi}c)(c-\xi)(1-|F|^2)\,\rho\left(\frac{1+\bar{a}c}{1-\bar{a}c}-p(c)\right)\operatorname{Re} f(\xi)}{[\{(1-\bar{\xi}c)(1+\bar{F})-(c-\xi)(1+F)\}\,p(c)+\{(1-\bar{\xi}c)(1-\bar{F})+(c-\xi)(1-F)\}]^2} + o(\rho^2) \quad \text{...(23)}$$

For this variation, in view of (17), we get

$$\delta \operatorname{Re} F(f(c)) = \rho \operatorname{Re}\left[\lambda X\left(\frac{1+\bar{a}c}{1-\bar{a}c}-p(c)\right)\right] + o(\rho^2). \quad \text{...(24)}$$

Since for maximum $\operatorname{Re} F(f(c))$, $\delta \operatorname{Re} F(f(c)) \leqslant 0$, on dividing by $\rho > 0$ and taking the limit as $\rho \to 0$ we obtain

$$\operatorname{Re}\left[\lambda X\left(p(c)-\frac{1+\bar{a}c}{1-\bar{a}c}\right)\right] \geqslant 0. \quad \text{...(25)}$$

As $\alpha \in E$ is arbitrary, letting $|\alpha| \to 1$ and putting $\alpha = z$, we have

$$B(z) = 2\operatorname{Re}\left[\bar{\lambda}\bar{X}\left(\overline{p(c)}-\frac{1+z\bar{c}}{1-z\bar{c}}\right)\right] \geqslant 0, \quad |z| = 1. \quad \text{...(26)}$$

As $B(z)$ is quadratic in z (has at least one zero on $|z| = 1$ and keeps the same sign on $|z| = 1$) $B(z)$ must have a second order zero on $|z| = 1$. Let this zero of $B(z)$ be denoted by $\bar{\eta}$, $|\eta| = 1$. Then, because $\operatorname{Re} p(z) > 0$ and has only one simple pole on $|z| = 1$,

$$p(z) = \frac{1+\eta z}{1-\eta z} = \frac{A(z)}{B(z)}. \quad \text{...(27)}$$

Consequently $A(z)$ and $B(z)$ must have a common factor $(1-\eta z)$ and we obtain from (21).

$$p(z) = \frac{G - 2iz \operatorname{Im} D - \bar{G}z^2}{G + 2z\operatorname{Re} D + \bar{G}z^2} = \frac{1-\eta^2 z^2}{(1-\eta z)^2}, \quad \text{...(28)}$$

where

$$\left.\begin{aligned} G &= c\,[\bar{\lambda}\bar{X}(1-\overline{p(c)}) - \lambda X(1+p(c))],\\ D &= \bar{\lambda}\bar{X}\,|c|^2(1+\overline{p(c)}) - \lambda X(1-p(c)).\end{aligned}\right\} \quad \text{...(29)}$$

Comparison of both sides of (28) yields

$$\operatorname{Im} G\eta = 0, \quad \text{...(30)}$$

$$\operatorname{Im} D = 0, \quad \text{...(31)}$$

$$G\eta = -D. \quad \text{...(32)}$$

Only two of these equations are independent. Equation (32) gives, when X has been replaced by (20) and G and D by (29).

$$p(c) = \frac{1 + c\eta}{1 - c\eta}, \qquad \text{...(33)}$$

and then (31) gives a quadratic in η the two roots of which are given by (15) and (16). One of these values gives the function $p(z)$ which when substituted in (6) gives max. Re $F(f(c))$. In view of Lemma 2 this value should give $B(z) \geqslant 0$ on $|z| = 1$ as can easily be verified.

Thus the proof of Theorem 3 is complete. In view of the importance of the corresponding result for the class P_1 we state the following:

Theorem 4—Let $c \in E$, $c \neq 0$, be fixed and let $F(w)$ be an analytic function of w in Re $w > 0$. If $f(z) \in P_1$ gives maximum or minimum Re $F(f(c))$ and $\lambda = F_w(f(c)) \neq 0$, then $f(z)$ has the form

$$f(z) = \frac{1 - a_1 z + (a_1 - z)\eta z}{1 + a_1 z + (a_1 + z)\eta z}, \quad |\eta| = 1. \qquad \text{...(34)}$$

If $f(z)$ gives the minimum, then

$$\eta = \frac{\bar{c}}{c} \frac{|\lambda|(1 - |c|^2)(1 + |c|^2 + 2a_1 \operatorname{Re} c) - 2i \operatorname{Im}[\lambda c(1 + a_1 \bar{c})(a_1 + \bar{c})]}{[\lambda(1 + a_1 \bar{c})^2 - \bar{\lambda}(a_1 + c)^2 \bar{c}^2]}, \qquad \text{...(35)}$$

and if $f(z)$ gives the maximum, then

$$\eta = -\frac{\bar{c}}{c} \frac{|\lambda|(1 - |c|^2)(1 + |c|^2 + 2a_1 \operatorname{Re} c) + 2i \operatorname{Im}[\lambda c(1 + a_1 \bar{c})(a_1 + \bar{c})]}{[\lambda(1 + a_1 \bar{c})^2 - \bar{\lambda}(a_1 + c)^2 \bar{c}^2]} \qquad \text{...(36)}$$

Theorem 5—If $f(z) \in P_1$ and $f'(0) = -2a_1$, $a_1 < 0$, then

$$\frac{|\xi f'(\xi)|}{\operatorname{Re} f(\xi)} \leqslant \frac{2|\xi|}{1 - |\xi|^2} \frac{2|\xi| - a_1(1 + |\xi|^2)}{1 - 2a_1|\xi| + |\xi|^2}. \qquad \text{...(37)}$$

The equality here is attained for the function

$$f(z) = \frac{1 - 2a_1 z + z^2}{1 - z^2} \qquad \text{...(38)}$$

at the point $z = |\xi|$.

PROOF: From (8) we have, on taking the limit as $z \to \xi$.

$$F(\xi, \xi) = \frac{f'(\xi)}{2 \operatorname{Re} f(\xi)}(1 - |\xi|^2). \qquad \text{...(39)}$$

In view of (10) we have

$$F(z, \xi) = \frac{\phi(z) + F(0, \xi)}{1 + \phi(z)\overline{F(0, \xi)}}, \quad \phi(0) = 0, \ |\phi(z)| \leqslant |z|. \qquad \text{...(40)}$$

and this gives

$$| F(\xi, \xi) | \leqslant \frac{| \phi(\xi) | + | F(0, \xi) |}{1 + | \phi(\xi) | | F(0, \xi) |}. \qquad ...(41)$$

Hence in order to prove the theorem we need to find an upper bound on $| F(0, \xi) |$. In view of (9) we obtain

$$| F(0, \xi) | = \frac{1}{| \xi |} \left| \frac{1 - f(\xi)}{1 + f(\xi)} \right| = \left| \frac{1}{\xi} \frac{1 - f(\xi)}{1 + f(\xi)} \right|. \qquad ...(42)$$

But

$$\frac{1}{\xi} \frac{1 - f(\xi)}{1 + f(\xi)}$$

is bounded analytic function in E whose value at $\xi = 0$ is a_1. Hence

$$| F(o, \xi) | = \left| \frac{1}{\xi} \frac{1 - f(\xi)}{1 + f(\xi)} \right| \leqslant \frac{| \xi | - a_1}{1 - a_1 | \xi |}. \qquad ...(43)$$

Therefore, from (41) we obtain in view of (43) and $| \phi(\xi) | \leqslant | \xi |$

$$| F(\xi, \xi) | \leqslant \frac{2 | \xi | - a_1 (1 + | \xi |^2)}{1 - 2a_1 | \xi | + | \xi |^2}. \qquad ...(44)$$

Taking the modulus of both sides in (39) and using (44) we obtain (37).

It may be remarked that (37) is a generalization of the well-known inequality (Robertson 1963).

$$\frac{| zf'(z) |}{\operatorname{Re} f(z)} \leqslant \frac{2 | z |}{1 - | z |^2},$$

for $f \in P$.

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