A NOTE ON QUASI-NORMAL OPERATORS

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An operator T on a Hilbert-space H is said to be normal if T commutes with T^*, quasi-normal if T commutes with T^*T (Brown 1953). The object of this paper is to study conditions on T which imply quasi normality. If T_1 and T_2 are two quasi-normal operators, we shall obtain conditions under which their product $T_1 T_2$ is quasi-normal.

Let $T = U + iV$, where $U = \text{Re } T = \frac{T + T^*}{2}$ and $V = \text{Im } T = \frac{T - T^*}{2i}$ are the real and imaginary parts of T. We shall write $B^2 = TT^*$ and $C^2 = T^*T$, where B and C are non-negative definite. Let $\sigma(T)$ denote the spectrum of T.

Firstly, we give necessary and sufficient conditions for an operator to be quasi-normal.

Theorem 1—T is quasi-normal if and only if C commutes with Re T and Im T.

PROOF: Let T be quasi-normal. Then $TT^*T = T^*T^2$ which implies $T^*TT^* = T^{*2}T$.

Now it is easy to see that $C^2 \text{ Re } T = \text{Re } TC^2$. Since C is non-negative definite, it follows that $C \text{ Re } T = \text{Re } TC$. Similarly $C \text{ Im } T = \text{Im } TC$.

Conversely, let $C \text{ Re } T = \text{Re } TC$ and $C \text{ Im } T = \text{Im } TC$. Then $C^2 \text{ Re } T = \text{Re } TC^2$ and $C^2 \text{ Im } T = \text{Im } TC^2$. Hence $C^2 [\text{Re } T + i \text{ Im } T] = [\text{Re } T + i \text{ Im } T] C^2$ and we have $C^2T = TC^2$. Therefore $T^*T^2 = TT^*T$.

In the following theorem we give conditions under which an operator T is quasi-normal.

Theorem 2—If T is an operator such that (*i*) B commutes with Re T and Im T (ii) $C^2T = TB^2$. Then T is quasi-normal.

PROOF: Since $B \text{ Re } T = \text{Re } TB$ and $B \text{ Im } T = \text{Im } TB$, we have,

$$B^2T + B^2T^* = TB^2 + T^*B^2,$$

$$B^2T - B^2T^* = TB^2 - T^*B^2.$$

This gives $B^2T = TB^2 = C^2T$. Hence T is quasi-normal.

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Now we prove the following:

Theorem 3—Let T be quasi-normal and $C^2T = TB^2$. Then B commutes with $\operatorname{Re} T$ and $\operatorname{Im} T$.

PROOF: Since T is quasi-normal and $C^2T = TB^2$, we have $T^*T^2 = T^2T^*$. Hence $T^{*2}T = TT^{*2}$.

Now

$$B^2 \operatorname{Re} T = \frac{TT^*T + TT^2}{2} = T^*T^2 + T^{*2}T = T^*T^2 + T^*TT^* = \operatorname{Re} T\, B^2.$$

Hence $B \operatorname{Re} T = \operatorname{Re} TB$. Similarly $B \operatorname{Im} T = \operatorname{Im} TB$.

We know every normal operator is quasi-normal but the converse is not true. In the following theorem we give the condition under which a quasi-normal operator is normal.

Theorem 4—Let T be quasi-normal. Then T is normal if $\sigma(T^*) \cap \sigma(T) = \phi$.

PROOF: Embry (1971) has proved that if $\sigma(T^*) \cap \sigma(T) = \phi$ then T^* and T commutes if and only if $T^* + T$ and T^*T commutes. Since T is quasi-normal therefore the result follows.

Theorem 5—Let T_1 and T_2 be two quasi-normal operators. Then their product T_1T_2 is quasi-normal if the following conditions are satisfied:

(i) $T_1T_2 = T_2T_1$ (ii) $T_1T_2^* = T_2^*T_1$

PROOF:

$$\begin{aligned}
& (T_1T_2)(T_1T_2)^*(T_1T_2) \\
&= (T_1T_2)(T_2^*T_1^*)(T_1T_2) \\
&= (T_1T_2)(T_1^*T_2^*)(T_1T_2) \\
&= T_1(T_2T_1^*)(T_2^*T_1)T_2 \\
&= T_1(T_1^*T_2)(T_1T_2^*)T_2 \\
&= T_1T_1^*(T_2T_1)(T_2^*T_2) \\
&= T_1T_1^*(T_1T_2)(T_2^*T_2) \\
&= (T_1^*T_1^2)(T_2^*T_2^2) \\
&= T_1^*(T_1^2T_2^*)T_2^2 \\
&= (T_1^*T_2^*)(T_1^2T_2^2) \\
&= (T_2^*T_1^*)(T_1T_2)^2 \\
&= (T_1T_2)^*(T_1T_2)^2
\end{aligned}$$

Hence T_1T_2 is quasi-normal.

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