SOME COUPLED THERMOELASTIC PROBLEMS CONCERNING ELASTIC HALF-SPACE

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(*Received* 9 *August* 1976)

The present paper studies the determination of thermal stress, deformation and temperature distribution in an elastic half-space taking into account the effect of coupling subject to the following mechanical and thermal boundary conditions: (i) the plane boundary is subjected to a step excitation of finite stress and is kept insulated; (ii) the plane boundary is subjected to a mechanical shock and is also kept insulated; and (iii) the plane boundary is rigidly fixed and is subjected to an instantaneous heat-flux.

The solutions obtained have the form of power series with respect to the coupling parameter ϵ. In determining the stress, deformation and temperature distribution terms up to linear ϵ (coupling constant) have been considered. The solutions are valid for arbitrary values of time t, provided ϵ is sufficiently small. In the last case the surface stress $\sigma(0, t_1)$ for various values of time t_1 is calculated and presented graphically.

INTRODUCTION

Hetnarski (1964 *a*, *b*) has determined stress and temperature distribution in an elastic half-space taking into account the effect of coupling of temperature and deformation fields in the cases when the plane boundary is maintained free of stress and subjected to a constant and instantaneous temperature distribution. Roychowdhury (1972) investigated a coupled thermoelastic problem of an elastic haf-space when the plane boundary was held rigidly fixed and subjected to a constant temperature applied for a finite interval. The present problem is concerned with the finding of stress, deformation and temperature distribution in elastic half-spaces taking into account the coupling of strain and temperature in the cases when the plane boundary is either rigidly fixed or subjected to a finite stress under various thermal conditions. The governing equations are solved by introducing thermoelastic potential. The Laplace transform is found useful as a convenient mathematical tool.

FORMULATION OF THE PROBLEM

We consider an elastic half-space $D: x \geqslant 0$. The space is assumed to be mechanically constrained so that the temperature and displacements of the form

$T=T(x,t)$, $U_x=U(x,t)$, $U_y=U_z=0$ occur, x and t denoting spatial co-ordinate and time respectively.

The coupled thermoelastic differential equations are

$$(\lambda+2\mu)\frac{\partial^2 U}{\partial x^2}=\rho\frac{\partial^2 U}{\partial t^2}+\alpha(3\lambda+2\mu)\frac{\partial T}{\partial x} \qquad ...(1)$$

$$\kappa\frac{\partial^2 T}{\partial x^2}=\rho C_v\cdot\frac{\partial T}{\partial t}+\alpha(3\lambda+2\mu)\overset{*}{T}\frac{\partial^2 U}{\partial x\partial t} \qquad ...(2)$$

where λ, μ are the isothermal Lamies constants,

ρ is the density of the material of the semi-space,

α the coefficient of linear expansion,

$\overset{*}{T}$ the reference temperature,

C_v the specific heat at constant volume, and

κ the thermal diffusivity.

The initial conditions are

$$(U)_{t=0}=\left(\frac{\partial U}{\partial t}\right)_{t=0}=0.$$

The thermal and boundary conditions are as follows:

Case I—Plane boundary is subjected to a step excitation of finite stress and the bounding surface is insulated, i.e.,

$$\sigma_{xx}=\sigma_0' H(t)$$

and

$$\frac{\partial T}{\partial x}=0$$

where

$$H(t)=1 \text{ for } t>0,\ H(t)=0 \text{ for } t<0,\ \sigma_0' \text{ is constant.}$$

Case II—Plane boundary is subjected to a mechanical shock and the bounding surface is insulated, i.e.,

$$\sigma_{xx}=\sigma_0'\,\delta(t) \text{ on } x=0$$

and

$$\frac{\partial T}{\partial x}=0.$$

Case III—Plane boundary is rigidly fixed and subjected to an instantaneous heat-flux, i.e.,

$$U=0 \text{ on } x=0$$

and

$$\frac{\partial T}{\partial x} = -T_0'\,\delta(t) \text{ where } T_0' \text{ is constant.}$$

We introduce the following dimensionless variables,

$$x_1 = \frac{ax}{\kappa}, \; t_1 = \frac{a^2 t}{\kappa}, \; \sigma = \frac{\sigma_{xx}}{\beta \overset{*}{T}},$$

$$T_1 = \frac{T}{\overset{*}{T}}, \; U_1 = \frac{a(\lambda + 2\mu)}{\kappa\beta\overset{*}{T}}\, U,$$

$$K = \frac{\kappa}{\rho C_v}, \; a^2 = \frac{\lambda + 2\mu}{\rho}, \; \beta = (3\lambda + 2\mu)\,\alpha.$$

Equations (1) and (2) then are transformed to

$$\frac{\partial^2 U_1}{\partial x_1^2} = \frac{\partial T_1}{\partial x_1} + \frac{\partial^2 U_1}{\partial t_1^2} \qquad \ldots(3)$$

$$\frac{\partial^2 T_1}{\partial x_1^2} = \frac{\partial T_1}{\partial t_1} + \varepsilon\, \frac{\partial^2 U_1}{\partial x_1\,\partial t_1} \qquad \ldots(4)$$

with the initial conditions

$$U_1, \; T_1, \frac{\partial U_1}{\partial t_1} = 0 \text{ at } t_1 = 0 \text{ for } x \geqslant 0$$

and the boundary conditions:

Case I: $\dfrac{\partial T_1}{\partial x_1} = 0, \; \sigma = \sigma_0\, H(t_1)$ on $x_1 = 0.$

Case II: $\dfrac{\partial T_1}{\partial x_1} = 0, \; \sigma = \sigma_0\, \delta(t_1)$ on $x_1 = 0.$

Case III: $U_1 = 0$ on $x_1 = 0$

$$\frac{\partial T_1}{\partial x_1} = -T_0\, \delta(t_1).$$

Also we have the regularity conditions

$$U_1, T_1, \frac{\partial U_1}{\partial x_1}, \; \frac{\partial T_1}{\partial x_1} \to 0 \text{ as } x_1 \to \infty.$$

The coupling parameter and the normal stress are, respectively, given below:

$$\varepsilon = \frac{\beta^2 \overset{*}{T}}{\rho C_v (\lambda + 2\mu)},$$

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial U}{\partial x} - (3\lambda + 2\mu)\,\alpha T = \beta \overset{*}{T}\left[\frac{\partial U_1}{\partial x_1} - T_1\right]. \qquad \ldots(5)$$

The non-dimensional stress will be

$$\sigma = \frac{\sigma_{xx}}{\beta \overset{*}{T}} = \frac{\partial U_1}{\partial x_1} - T_1. \qquad ...(6)$$

We introduce the thermoelastic potential ϕ in dimensionless form such that

$$U_1 = \frac{\partial \phi}{\partial x_1}.$$

Equation (3) then reduces to

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial t_1^2}\right) \frac{\partial \phi}{\partial x_1} = \frac{\partial T_1}{\partial x_1} \qquad ...(7)$$

which is identically satisfied if

$$T_1 = \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial t_1^2}\right) \phi + X(t_1). \qquad ...(7a)$$

Equation (4) thus reduces to

$$\left[\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial t_1}\right)\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial t_1^2}\right) - \varepsilon \frac{\partial^3}{\partial x_1^2 \partial t_1}\right] \phi = 0$$

i.e.,

$$\left[\frac{\partial^4}{\partial x_1^4} - \frac{\partial^3}{\partial x_1^2 \partial t_1}\left(\frac{\partial}{\partial t_1} + 1 + \varepsilon\right) + \frac{\partial^3}{\partial t_1^3}\right] \phi = 0 \qquad ...(8)$$

abiding by the initial conditions

$$\phi = \frac{\partial \phi}{\partial t_1} = \frac{\partial^2 \phi}{\partial t_1^2} = 0 \text{ at } t_1 = 0.$$

Also,

$$\sigma = \frac{\partial U_1}{\partial x_1} - T_1 = \frac{\partial^2 \phi}{\partial x_1^2} - \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial t_1^2}\right) \phi = \frac{\partial^2 \phi}{\partial t_1^2}.$$

The Laplace transform of $\phi(x_1, t_1)$ is given by

$$\bar{\phi}(x_1, p) = L[\phi] = \int_0^\infty \phi(x_1, t_1) e^{-pt_1} dt_1.$$

Therefore, the Laplace transform of (8) will give

$$\frac{d^4 \bar{\phi}}{dx_1^4} - (p + 1 + \varepsilon) p \frac{d^2 \bar{\phi}}{dx_1^2} + p^3 \bar{\phi} = 0. \qquad ...(9)$$

The general solution of (9) vanishing at $x_1 = \infty$ is of the form

$$\bar{\phi}(x_1 p) = Ae^{-m_1 x_1} + Be^{-m_2 x_1} \qquad ...(10)$$

where A and B are constants to be determined from the loading and heating conditions at $x_1 = 0$. The quantities m_1^2 and m_2^2 are the roots of the quadratic (in m^2)

$$m^4 - p(1 + p + \varepsilon) m^2 + p^3 = 0$$

and

$$m^2_{1,2} = \frac{p}{2}\{p + 1 + \varepsilon \pm [p^2 - 2(1-\varepsilon)p + (1+\varepsilon)^2]^{\frac{1}{2}}\}. \qquad \ldots(11)$$

It can be seen that

$$m_1(p, 0) = (m_1)_{\epsilon=0} = p,$$

$$m_2(p, 0) = (m_2)_{\epsilon=0} = \sqrt{p},$$

$$\left[\frac{\partial m_1}{\partial \varepsilon}\right]_{\epsilon=0} = \frac{p}{2(p-1)}, \quad \left[\frac{\partial m_2}{\partial \varepsilon}\right]_{\epsilon=0} = \frac{\sqrt{p}}{2(p-1)},$$

$$\left[\frac{\partial^2 m_1}{\partial \varepsilon^2}\right]_{\epsilon=0} = -\frac{p(p+3)}{4(p-1)^3}, \quad \left[\frac{\partial^2 m_2}{\partial \varepsilon^2}\right]_{\epsilon=0} = \frac{\sqrt{p}(3p+1)}{4(p-1)^3}, \quad \text{etc.}$$

The expressions for temperature and stress in the domain of Laplace transform are

$$\bar{T}_1 = \left(\frac{d^2}{dx_1^2} - p^2\right)\bar{\phi}(x_1, p),$$

$$\bar{\sigma} = p^2\bar{\phi}(x_1, p). \qquad \cdots(12)$$

Case I

Boundary conditions yield

$$\left.\begin{aligned} &\bar{\phi} = \frac{\sigma_0}{p^3} \\ \text{and} \\ &\frac{d^3\bar{\phi}}{dx_1^3} - p^2\frac{d\bar{\phi}}{dx_1} = 0 \end{aligned}\right\} \text{ on } x_1 = 0.$$

The above conditions give

$$A + B = \frac{\sigma_0}{p^3}$$

and

$$Am_1^3 + Bm_2^3 - p^2(Am_1 + Bm_2) = 0.$$

The solution of these two equations yield

$$A = \frac{\sigma_0 m_2(p^2 - m_2^2)}{p^3(m_1 - m_2)(m_1^2 + m_2^2 + m_1m_2 - p^2)}$$

$$B = \frac{\sigma_0 m_1(p^2 - m_1^2)}{p^3(m_2 - m_1)(m_1^2 + m_2^2 + m_1m_2 - p^2)},$$

Hence,

$$\bar{\phi}(x_1, p; \varepsilon) = \frac{\sigma_0}{p^3} \times \frac{\{m_2(p^2 - m_2^2)e^{-m_1x_1} - m_1(p^2 - m_1^2)e^{-m_2x_1}\}}{(m_1 - m_2)(m_1^2 + m_2^2 + m_1m_2 - p^2)}$$

$$\bar{\sigma}(x_1, p; \varepsilon) = \frac{\sigma_0}{p} \times \frac{\{m_2(p^2 - m_2^2)e^{-m_1x_1} - m_1(p^2 - m_1^2)e^{-m_2x_1}\}}{(m_1 - m_2)(m_1^2 + m_2^2 + m_1m_2 - p^2)}$$

$$\bar{T}_1(x_1, p; \varepsilon) = \frac{\sigma_0}{p^3} \times \frac{m_2(p^2 - m_1^2)(p^2 - m_2^2)\{m_1 . e^{-m_2x_1} - m_2 e^{-m_1x_1}\}}{(m_1 - m_2)(m_1^2 + m_2^2 + m_1m_2 - p^2)},$$

and

$$\bar{U}_1(x_1, p; \varepsilon) = \frac{\sigma_0}{p^3} \times \frac{m_1m_2[(p^2 - m_1^2)e^{-m_2x_1} - (p^2 - m_2^2)e^{-m_1x_1}]}{(m_1 - m_2)(m_1^2 + m_2^2 + m_1m_2 - p^2)}. \qquad ...(13)$$

On expanding these expressions for stress, temperature and deformation into a power series in the neighbourhood of $\varepsilon = 0$ we get,

$$\bar{T}_1(x_1, p; \varepsilon) = \bar{T}_1(x_1, p; 0) + \frac{\varepsilon}{1!}\left[\frac{\partial \bar{T}_1}{\partial \varepsilon}(x_1, p; \varepsilon)\right]_{\varepsilon=0} + \frac{\varepsilon^2}{2!}\left[\frac{\partial^2 \bar{T}_1}{\partial \varepsilon^2}\right]_{\varepsilon=0} + \cdots$$

$$\bar{\sigma}(x_1, p; \varepsilon) = \bar{\sigma}(x_1, p; 0) + \frac{\varepsilon}{1!}\left[\frac{\partial \bar{\sigma}}{\partial \varepsilon}(x_1, p; \varepsilon)\right]_{\varepsilon=0} + \frac{\varepsilon^2}{2!}\left[\frac{\partial^2 \bar{\sigma}}{\partial \varepsilon^2}\right]_{\varepsilon=0} + \cdots$$

$$\bar{U}_1(x_1; p; \varepsilon)_{x_1=0} = \bar{U}_1(0, p; 0) + \frac{\varepsilon}{1!}\left[\frac{\partial \bar{U}_1}{\partial \varepsilon}(o, p; \varepsilon)\right]_{\varepsilon=0} + \frac{\varepsilon^2}{2!}\left[\frac{\partial^2 \bar{U}_1}{\partial \varepsilon^2}\right]_{\varepsilon=0} + \cdots$$

Now

$$\left[\frac{\partial \bar{T}_1}{\partial \varepsilon}\right]_{\varepsilon=0} = \sigma_0\left[\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(p-1)} - \frac{e^{-px_1}}{p(p-1)}\right].$$

Keeping terms up to linear ε,

$$\bar{T}_1(x_1, p; \varepsilon) = \sigma_0\varepsilon\left[\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(p-1)} - \frac{e^{-px_1}}{p(p-1)}\right].$$

Again,

$$\left[\frac{\partial \bar{\sigma}}{\partial \varepsilon}\right]_{\varepsilon=0} = \frac{\sigma_0\,\varepsilon}{\sqrt{p}(p-1)^2}\left[pe^{-\sqrt{p}x_1} - pe^{-px_1} - \frac{\sqrt{p}(p-1)x_1}{2}e^{-px_1}\right]$$

$$\therefore\ \bar{\sigma}(x_1, p : \varepsilon) = \frac{\sigma_0 e^{-px_1}}{p} + \frac{\sigma_0\varepsilon}{\sqrt{p}(p-1)^2}\left[pe^{-\sqrt{p}x_1} - pe^{-px_1} - \frac{\sqrt{p}(p-1)x_1}{2}e^{-px_1}\right].$$

On $x_1 = 0$, $\bar{U}_1(0, p; \varepsilon) = \frac{\sigma_0}{p^3} \times \frac{(m_1 + m_2)m_1m_2}{(p^2 - m_1^2 - m_2^2 - m_1m_2)}$

$$\therefore\ \bar{U}_1(0, p; \varepsilon) = -\frac{\sigma_0}{p^2} + \frac{\sigma_0\,\varepsilon}{p^2} \times \frac{\sqrt{p} + \frac{1}{2}}{(\sqrt{p} + 1)^2}.$$

Restricting the results to small values of time we can expand the expressions for $\bar{U}_1$ in power series of $1/p$ which give rise to

$$\bar{U}_1\ (o, p; \varepsilon) = -\frac{\sigma_0}{p^2} + \frac{\sigma_0\varepsilon}{p^{5/2}} - \frac{3\sigma_0\varepsilon}{2p^3} + \frac{2\sigma_0\varepsilon}{p^{7/2}} + \frac{3\sigma_0\varepsilon}{2p^4}$$

Inversions

$$U_1\,(o, t_1; \varepsilon) = -\sigma_0 t_1 + \frac{4\sigma_0\varepsilon\, t_1^{3/2}}{3\sqrt{\pi}} - \frac{3\sigma_0\varepsilon t_1^{2}}{4} + \frac{16\sigma_0\varepsilon\, t_1^{5/2}}{15\sqrt{\pi}} + \frac{\sigma_0\varepsilon t_1^{3}}{4}$$

for small values of time.

Now,

$$L^{-1}\left[\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}\,(p-1)}\right] = \frac{e^{t_1}}{2}\left[e^{-x_1}\operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} - \sqrt{t_1}\right)\right.$$
$$\left. - e^{x_1}\operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} + \sqrt{t_1}\right)\right] = V(x_1, t_1), \text{ say.}$$

$$L^{-1}\left[\frac{e^{-px_1}}{p\,(p-1)}\right] = e^{t_1-x_1}\,H(t_1-x_1) - H(t_1-x_1)$$

$$\therefore\quad T_1(x_1, t_1) = \frac{\sigma_0\varepsilon\, e^{t_1}}{2}\left[e^{-x_1}\operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} - \sqrt{t_1}\right)\right.$$
$$\left. - e^{x_1}\operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} + \sqrt{t_1}\right)\right] - \sigma_0\varepsilon\,\{e^{t_1-x_1}\,H(t_1-x_1)$$
$$- H(t_1-x_1)\}.$$

Again,

$$L^{-1}\left[\frac{e^{-\sqrt{p}x_1}}{(p-1)}\right] = \frac{e^{t_1}}{2}\left[e^{-x_1}\operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} - \sqrt{t_1}\right)\right.$$
$$\left. + e^{x_1}\operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} + \sqrt{t_1}\right)\right] = U(x_1, t_1),$$

$$L^{-1}\left[\frac{\sqrt{p}\,e^{-\sqrt{p}x_1}}{(p-1)^2}\right] = -\frac{x_1}{2}\,U + \left(t_1 + \frac{1}{2}\right)V + \sqrt{\frac{t_1}{\pi}}\,e^{-x_1^2/4t_1}$$

$$L^{-1}\left[\frac{\sqrt{p}\,e^{-px_1}}{(p-1)^2}\right] = L^{-1}\left[\frac{e^{-px_1}}{\sqrt{p}\,(p-1)} + \frac{e^{-px_1}}{\sqrt{p}\,(p-1)^2}\right]$$
$$= e^{t_1-x_1}\ \operatorname{erf}\ (\sqrt{t_1-x_1})\cdot H(t_1-x_1)$$
$$+ \left\{\frac{t_1-x_1}{2}\,e^{t_1-x_1}\operatorname{erf}(\sqrt{t_1-x_1}) + \sqrt{\frac{t_1-x_1}{\pi}}\right\}$$
$$\times H(t_1-x_1).$$

$$\therefore\quad \sigma(x_1, t_1) = \sigma_0 H(t_1 - x_1) + \sigma_0\varepsilon\left[-\frac{x_1}{2}\,U + \left(t_1 + \frac{1}{2}\right)V\right.$$
$$+ \sqrt{\frac{t_1}{\pi}}\,e^{-(x_1^2/4t_1)} - e^{t_1-x_1}\ \operatorname{erf}\ (\sqrt{t_1-x_1})\,H(t_1-x_1) -$$

(*equation continued on p.* 473)

$$-\left\{\frac{t_1-x_1}{2}e^{t_1-x_1}\ \mathrm{erf}\ (\sqrt{t_1-x_1})+\frac{\sqrt{t_1-x_1}}{\sqrt{\pi}}\right\}H(t_1-x_1)$$

$$-\frac{x_1}{2}e^{t_1-x_1}\Bigg].$$

Case II

The boundary conditions yield

$$\bar{\phi}=\frac{\sigma_0}{p^2}\ \text{on}\ x_1=0$$

and

$$\frac{d^3\bar{\phi}}{dx_1^{\,3}}-p^2\frac{d\bar{\phi}}{dx_1}=0\ \text{on}\ x_1=0.$$

Using these conditions,

$$A=\frac{\sigma_0}{p^2}\times\frac{m_2(m_2^{\,2}-p^2)}{(m_2^{\,3}-m_1^{\,3}-m_2p^2+m_1p^2)}$$

$$B=\frac{\sigma_0}{p^2}\times\frac{m_1(m_1^{\,2}-p^2)}{(m_2^{\,3}-m_1^{\,3}-m_1p^2+m_2p^2)}$$

$$\therefore\quad \bar{\phi}(x_1,p;\varepsilon)=\frac{\sigma_0}{p^2}\left[\frac{m_2(p^2-m_2^{\,2})e^{-m_1x_1}-m_1(p^2-m_1^{\,2})e^{-m_2x_1}}{(m_1-m_2)(m_1^{\,2}+m_2^{\,2}+m_1m_2-p^2)}\right]$$

$$\bar{\sigma}(x_1,p;\varepsilon)=\sigma_0\left[\frac{m_2(p^2-m_2^{\,2})e^{-m_1x_1}-m_1(p^2-m_1^{\,2})e^{-m_2x_1}}{(m_1-m_2)(m_1^{\,2}+m_2^{\,2}+m_1m_2-p^2)}\right]$$

$$\bar{T}_1(x_1,p;\varepsilon)=\frac{\sigma_0}{p^2}\times\frac{(m_1^{\,2}-p^2)(m_2^{\,2}-p^2)\{m_1e^{-m_2x_1}-m_2e^{-m_1x_1}\}}{m_2(p^2-m_2^{\,2})-m_1(p^2-m_1^{\,2})}$$

and

$$\bar{U}_1(x_1,p;\varepsilon)$$

$$=\frac{\sigma_0}{p^2}\times\frac{m_1m_2[(p^2-m_1^{\,2})e^{-m_2x_1}-(p^2-m_2^{\,2})e^{-m_1x_1}]}{(m_1-m_2)(m_1^{\,2}+m_2^{\,2}+m_1m_2-p^2)}\ .$$

Expanding the above expressions for $\bar{\sigma}$, $\bar{T}_1$ and $\bar{U}_1$ into power series in the neighbourhood of $\varepsilon=0$ and considering terms upto linear ε (which is very small), we have,

$$\bar{T}_1(x_1,p;\varepsilon)=\sigma_0\varepsilon\left[\frac{\sqrt{p}e^{-\sqrt{p}x_1}}{(p-1)}-\frac{e^{-px_1}}{(p-1)}\right]$$

$$\bar{\sigma}(x_1,p;\varepsilon)=\sigma_0e^{-px_1}+\frac{\sigma_0\varepsilon\sqrt{p}}{(p-1)^2}\left[pe^{-\sqrt{p}x_1}-pe^{-px_1}-\frac{\sqrt{p}(p-1)}{2}x_1e^{-px_1}\right].$$

On the plane boundary,

$$\bar{U}_1(o,p;\varepsilon)=-\frac{\sigma_0}{p}+\frac{\sigma_0\varepsilon}{p}\times\frac{\sqrt{p}+\frac{1}{2}}{(\sqrt{p}+1)^2}\ .$$

For small values of time, the expression for $\bar{U}_1(o, p; \varepsilon)$ on expansion in powers of $1/p$ reduces to

$$\bar{U}_1(o, p; \varepsilon) = -\frac{\sigma_0}{p} + \frac{\sigma_0\varepsilon}{p^{3/2}} - \frac{3\sigma_0\varepsilon}{2p^2} + \frac{2\sigma_0\varepsilon}{p^{5/2}} + \frac{3\sigma_0\varepsilon}{2p^3}.$$

Inversions—Then,

$$U_1(o, t_1) = -\sigma_0 + \frac{2\sigma_0\varepsilon\sqrt{t_1}}{\sqrt{\pi}} - \frac{3\sigma_0 t_1}{2} + \frac{8\sigma_0\varepsilon t_1^{3/2}}{3} + \frac{3\sigma_0\varepsilon t_1^2}{4}.$$

Now

$$L^{-1}\left[\frac{\sqrt{p}e^{-\sqrt{p}x_1}}{p-1}\right] = L^{-1}\left[\frac{(p-1+1)e^{-\sqrt{p}x_1}}{\sqrt{p}(p-1)}\right] = L^{-1}\left[\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}} + \frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(p-1)}\right]$$

$$= \frac{e^{-(x_1^2/4t_1)}}{\sqrt{\pi t_1}} + V(x_1, t_1).$$

$$\therefore T_1(x_1, t_1) = \sigma_0\varepsilon\left[\frac{e^{-(x_1^2/4t_1)}}{\sqrt{\pi t_1}} + V(x_1, t_1) - e^{t_1 - x_1} H(t_1 - x_1)\right]$$

$$L^{-1}\left[\frac{p\sqrt{p}e^{-\sqrt{p}x_1}}{(p-1)^2}\right] = L^{-1}\left[\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}} + \frac{2\sqrt{p}e^{-\sqrt{p}x_1}}{(p-1)^2} - \frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(p-1)^2}\right]$$

$$L^{-1}\left[\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(p-1)^2}\right] = L^{-1}\left[\frac{1}{4}\cdot\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(\sqrt{p}-1)^2} + \frac{1}{4}\cdot\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(\sqrt{p}+1)^2}\right.$$

$$\left. - \frac{1}{4}\cdot\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(\sqrt{p}-1)} + \frac{1}{4}\cdot\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(\sqrt{p}+1)}\right].$$

We know

$$L^{-1}\left[\frac{e^{-\alpha\sqrt{p}}}{\sqrt{p}(\sqrt{p}+\beta)^2}\right] = 2\sqrt{\frac{t_1}{\pi}}\exp\left(-\frac{\alpha^2}{4t_1}\right) - (2\beta t_1 + \alpha)\exp(\alpha\beta + \beta^2 t_1)$$

$$\times \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{t_1}} + \beta\sqrt{t_1}\right).$$

$$L^{-1}\left[\frac{e^{-\alpha\sqrt{p}}}{\sqrt{p}(\sqrt{p}+\beta)}\right] = \exp\{\beta(\beta t_1 + \alpha)\}\operatorname{erfc}\left(\beta\sqrt{t_1} + \frac{\alpha}{2\sqrt{t_1}}\right).$$

Hence

$$L^{-1}\left[\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(p-1)^2}\right]$$

$$= \frac{1}{4}\left[2\sqrt{\frac{t_1}{\pi}}\exp\left(-\frac{x_1^2}{4t_1}\right) - (x_1 - 2t_1)\exp(t_1 - x_1)\cdot\right.$$

$$\times \operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} - \sqrt{t_1}\right) + 2\sqrt{\frac{t_1}{\pi}}\exp\left(-\frac{x_1^2}{4t_1}\right) - (x_1 + 2t_1)$$

$$\times \exp(t_1 + x_1) \cdot \operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} + \sqrt{t_1}\right) - e^{t_1 - x_1} \operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} - \sqrt{t_1}\right)$$

$$+ e^{t_1 - x_1} \operatorname{erfc}\left(\frac{x_1}{2\sqrt{t_1}} + \sqrt{t_1}\right) = W(x_1, t_1), \text{ (say)}$$

$$\therefore \bar{\sigma}(x_1, p) = \sigma_0 e^{-px_1} + \sigma_0 \varepsilon \left[\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}} + \frac{2\sqrt{p}e^{-\sqrt{p}x_1}}{(p-1)^2} - \frac{e^{-\sqrt{p}x_1}}{\sqrt{p}(p-1)^2} \right.$$

$$- \left\{ \frac{e^{-px_1}}{\sqrt{p}} + \frac{e^{-px_1}}{\sqrt{p}(p-1)} + \frac{\sqrt{p}e^{-px_1}}{(p-1)^2} \right\}$$

$$\left. - \frac{x_1}{2} \left\{ \left(e^{-px_1} - \frac{e^{-px_1}}{p-1} \right) \right\} \right].$$

Since,

$$L^{-1}\left[\frac{e^{-px_1}}{\sqrt{p}(p-1)} \right] = e^{t_1 - x_1} \operatorname{erf}(\sqrt{t_1 - x_1})\, H(t_1 - x_1)$$

$$L^{-1}\left[\frac{\sqrt{p}e^{-px_1}}{(p-1)^2} \right] = L^{-1}\left[\frac{e^{-px_1}}{\sqrt{p}(p-1)} \right] + L^{-1}\left[\frac{e^{-px_1}}{\sqrt{p}(p-1)^2} \right]$$

$$= e^{t_1 - x_1} \operatorname{erf}(\sqrt{t_1 - x_1})\, H(t_1 - x_1) + \left\{ \frac{t_1 - x_1}{2} e^{t_1 - x_1} \cdot \operatorname{erf}(\sqrt{t_1 - x_1}) \right.$$

$$\left. + \sqrt{\frac{t_1 - x_1}{\pi}} \right\} \cdot H(t_1 - x_1)$$

$$\sigma(x_1, t_1) = \sigma_0\, \delta(t_1 - x_1) + \sigma_0 \varepsilon \left[\frac{e^{-x_1^2/4t_1}}{\sqrt{\pi t_1}} - x_1 U(x_1, t_1) \right.$$

$$+ (2t_1 + 1)\, V(x_1, t_1) + 2\sqrt{\frac{t_1}{\pi}} \exp\left(-\frac{x_1^2}{4t_1}\right) - W(x_1, t_1)$$

$$- \frac{H(t_1 - x_1)}{\sqrt{\pi(t_1 - x_1)}} - 2e^{t_1 - x_1} \operatorname{erf}(\sqrt{(t_1 - x_1)})\, H(t_1 - x_1)$$

$$- \left\{ \frac{t_1 - x_1}{2} e^{t_1 - x_1} \operatorname{erf}(\sqrt{t_1 - x_1}) + \frac{\sqrt{t_1 - x_1}}{\pi} \right\}$$

$$\times H(t_1 - x_1) - \frac{x_1}{2} \delta(t_1 - x_1) - e^{t_1 - x_1} \frac{x_1}{2} H(t_1 - x_1).$$

Case III

The boundary conditions give

$d\bar{\phi}/dx_1 = 0.$

and

$$\frac{d^3 \bar{\phi}}{dx_1^3} - p^2 \frac{d\bar{\phi}}{dx_1} = -T_0 \text{ at } x_1 = 0.$$

Substituting the value of $\bar{\phi}$ in the above conditions and solving for A and B, we have

$$A = \frac{T_0}{m_1 (m_1^2 - m_2^2)}$$

$$B = \frac{-T_0}{m_2 (m_1^2 - m_2^2)} .$$

$$\therefore \bar{\phi} (x_1 p; \varepsilon) = T_0 \times \frac{(m_1^2 e^{-m_2 x_1} - m_2^2 e^{-m_1 x_1})}{m_1 m_2 \{m_2 (p^2 - m_1^2) - m_1 (p^2 - m_2^2)\}}$$

$$\bar{\sigma} (x_1, p; \varepsilon) = T_0 p^2 \times \frac{m_1^2 e^{-m_2 x_1} - m_2^2 e^{-m_1 x_1}}{m_1 m_2 \{m_2 (p^2 - m_1^2) - m_1 (p^2 - m_2^2)\}}$$

$$\bar{T_1} (x_1, p; \varepsilon) = T_0 \times \frac{\{m_1^2 (m_2^2 - p^2) e^{-m_2 x_1} - m_2^2 (m_1^2 - p^2) e^{-m_1 x_1}\}}{m_1 m_2 \{m_2 (p^2 - m_1^2) - m_1 (p^2 - m_2^2)\}} .$$

Expanding these expressions into power series in the neighbourhood of $\varepsilon = 0$ and retaining terms upto first power of ε, we have,

$$\bar{\sigma} (x_1, p; \varepsilon) = T_0 \left(\frac{e^{-p x_1}}{\sqrt{p} (p-1)} - \frac{\sqrt{p} e^{-\sqrt{p} x_1}}{(p-1)} \right) - \frac{T_0 \varepsilon}{p^2 (p-1)^2} \left[\frac{p^2 \sqrt{p} x_1}{2} \right.$$

$$\times e^{-p x_1} - p \sqrt{p} e^{-p x_1} + \frac{p^3 x_1}{2} e^{-\sqrt{p} x_1} + p^2 \sqrt{p} e^{-\sqrt{p} x_1}$$

$$\left. - \left(\frac{e^{-p x_1}}{\sqrt{p}} - \sqrt{p} e^{-\sqrt{p} x_1} \right) \left(\frac{p^2}{2} - \frac{p^2}{p-1} - \frac{p^2 \sqrt{p}}{p-1} \right) \right].$$

Inversion

As the inversions are much complicated, we consider surface stress distribution only, for $t_1 > 0$

$$\bar{\sigma} (0, p ; \varepsilon) = -\frac{T_0}{\sqrt{p}} - T_0 \varepsilon \left[\frac{1}{2\sqrt{p}} + \frac{1}{2} \cdot \frac{1}{\sqrt{p} (p-1)} - \frac{1}{\sqrt{p} (p-1)^2} \right.$$

$$\left. - \frac{1}{p-1} - \frac{1}{(p-1)^2} \right].$$

Now, $$L^{-1} \left[\frac{1}{\sqrt{p}} \right] = \frac{1}{\sqrt{\pi t_1}} , \quad L^{-1} \left[\frac{1}{\sqrt{p} (p-1)} \right] = e^{t_1} \operatorname{erf} (\sqrt{t_1})$$

$$L^{-1} \left[\frac{1}{\sqrt{p} (p-1)^2} \right] = \frac{t_1 e^{t_1}}{2} \operatorname{erf} (\sqrt{t_1}) + \sqrt{\frac{t_1}{\pi}} .$$

Hence,

$$\sigma (0, t_1) = -\frac{T_0}{\sqrt{\pi t_1}} - T_0 \varepsilon \left[\frac{1}{2} \cdot \frac{1}{\sqrt{\pi t_1}} + \frac{e^{t_1}}{2} \operatorname{erf} (\sqrt{t_1}) \right.$$

$$\left. - \frac{t_1 e^{t_1}}{2} \operatorname{erf} (\sqrt{t_1}) - \sqrt{\frac{t_1}{\pi}} - (t_1 + 1) e^{t_1} \right]$$

$$\therefore -\frac{2\sigma(o,t_1)\sqrt{\pi}}{T_0} = \frac{1}{\sqrt{t_1}} + \varepsilon\Big[\frac{1}{\sqrt{t_1}} + \sqrt{\pi}e^{t_1}\operatorname{erf}(\sqrt{t_1})$$
$$-\sqrt{\pi t_1}\,.\,e^{t_1}\operatorname{erf}(\sqrt{t_1}) - 2\sqrt{t_1} - 2\sqrt{\pi}e^{t_1}(t_1+1)\Big].$$

Discussion

The surface stress distribution $\sigma(0, t_1)$ in the last case for various values of time t_1 is calculated and shown in Fig. 1. From Fig. 1, it is observed that surface stress very near to the initial instant is very large and it decays very rapidly with time. This is so because in this case the plane surface is subjected to an instantaneous heat-flux.

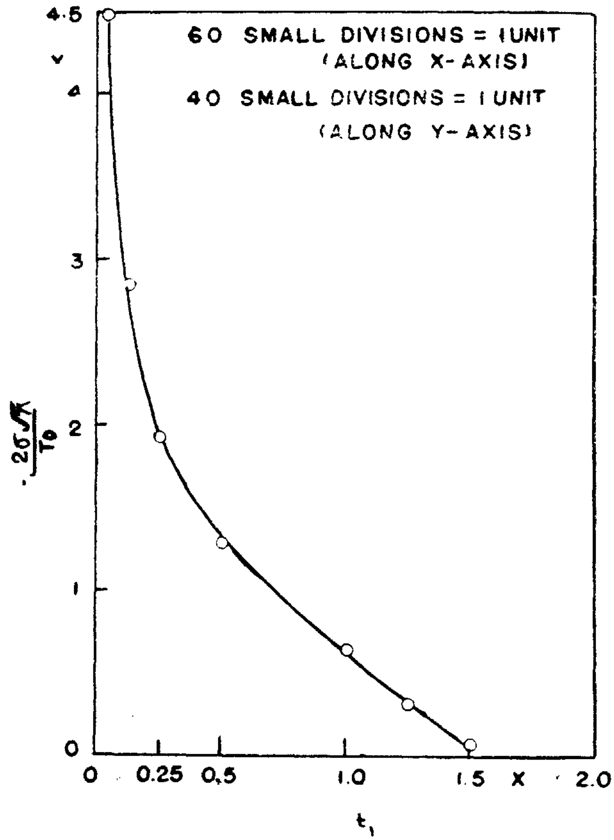

Fig. 1

Acknowledgement

The author expresses his grateful thanks to Dr. S. K. Roy Chaudhuri, Department of Mathematics, Jadavpur University, for his active help in preparing the paper.

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