ON THREE FIXED POINT MAPPINGS FOR COMPACT METRIC SPACES

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It is proved that if T is a continuous mapping of a compact metric space X into itself, satisfying the inequality

$$\rho(Tx, Ty) < a\rho(x, y) + b\{\rho(x, Tx) + \rho(y, Ty)\} + c\{\rho(x, Ty) + \rho(y, Tx)\}$$

for all distinct x, y in X, where

$$a + 2(b + c) = 1, \quad b + c < 1, \quad a + 2c \leqslant 1, \quad c \geqslant 0,$$

then T has a unique fixed point.

In a paper by Edelstein (1962), he proves the following theorem:

Theorem 1—If T is a mapping of the compact metric space X into itself, satisfying the inequality

$$\rho(Tx, Ty) < \rho(x, y)$$

for all distinct x, y in X, then T has a unique fixed point.

More recently the author (Fisher 1976*a*, *b*) has proved the following theorems:

Theorem 2—If T is a continuous mapping of a compact metric space X into itself, satisfying the inequality

$$\rho(Tx, Ty) < \tfrac{1}{2}\{\rho(x, Tx) + \rho(y, Ty)\}$$

for all distinct x, y in X, then T has a unique fixed point.

Theorem 3—If T is a continuous mapping of a compact metric space X into itself, satisfying the inequality

$$\rho(Tx, Ty) < \tfrac{1}{2}\{\rho(x, Ty) + \rho(y, Tx)\}$$

for all distinct x, y in X, then T has a unique fixed point.

We will now prove the following theorem which includes each of these theorems as special cases:

Theorem 4—If T is a continuous mapping of a compact metric space X into itself, satisfying the inequality

$$\rho(Tx, Ty) < a\rho(x, y) + b\{\rho(x, Tx) + \rho(y, Ty)\} + c\{\rho(x, Ty) + \rho(y, Tx)\}$$

or all distinct x, y in X, where

$$a + 2(b + c) = 1,\ b + c < 1,\ a + 2c \leqslant 1,\ c \geqslant 0,$$

then T has a unique fixed point.

PROOF: Define a function f on X by

$$f(x) = \rho(x, Tx)$$

for all x in X. Since ρ and T are continuous functions, it follows that f is a continuous function on X. Since X is compact there exists a point z in X such that

$$f(z) = \inf\{f(x) : x \in X\}.$$

If $f(z) \neq 0$, it follows that $Tz \neq z$ and so

$$\begin{aligned} f(Tz) &= \rho(Tz, T^2 z) \\ &< a\rho\{(z, Tz) + b\{\rho(z, Tz) + \rho(Tz, T^2 z)\} + c\rho(z, T^2 z) \\ &\leqslant (a + b + c)\rho(z, Tz) + (b + c)\rho(Tz, T^2 z), \end{aligned}$$

since $c \geqslant 0$. It follows that since $b + c < 1$,

$$\begin{aligned} \rho(Tz, T^2 z) &< \frac{a + b + c}{1 - b - c}\rho(z, Tz) \\ &= \rho(z, Tz) \end{aligned}$$

or

$$f(Tz) < f(z).$$

This contradicts the definition of z so that we must have $Tz = z$ and z is then a fixed point of T.

Now suppose that T has a second distinct fixed point z'. Then

$$\begin{aligned} \rho(z, z') &= \rho(Tz, Tz') \\ &< a\rho(z, z') + b\{\rho(z, Tz) + \rho(z', Tz')\} + c\{\rho(z, Tz') + \rho(z', Tz)\} \\ &= (a + 2c)\rho(z, z'). \end{aligned}$$

giving a contradiction since $a + 2c \leqslant 1$. It follows that the fixed point is unique, completing the proof of the theorem.

We finally note that if $b = c = 0$, then $a = 1$ which gives Theorem 1, if $a = c = 0$, then $b = \frac{1}{2}$ which gives Theorem 2, and if $a = b = 0$, then $c = \frac{1}{2}$ which gives Theorem 3.

References

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