A CONTRIBUTION TO THE NONHOLONOMIC MECHANICAL SYSTEMS OF THE SECOND ORDER

by Dj. S. Djukic, *Faculty of Technical Sciences, University of Novi Sad,* 21000 *Novi Sad, Yugoslavia*

(*Received* 15 *July* 1976)

A geometrization of motion of the nonholonomic mechanical systems of the second order is given. Also, a connection between the motion of nonconservative mechanical systems and the motion of nonholonomic systems is established.

§1. Mutual actions of different physical natures than those considered in classical mechanics have been discussed by Dobronravov (1961). These mechanical systems are usually subject to the nonholonomic constraints of the second and higher order. A review of the literature devoted to the nonholonomic problems of higher order may be found in Djukic (1972).

Under the concept of the geometrization of motion we consider the finding of a space in which the differential equations of motion are identical with the geodesic lines of the space. Subsequent to the findings of Synge (1926) and Douglas (1928) attempts were made to describe the motion of nonconservative mechanical systems and mechanical systems with nonholonomic constraints of the first order with the help of non-Riemannian geometry (see Djukic and Vujanovic 1975).

Here, a geometrization of the motion of mechanical systems subject to linear nonholonomic constraints of the second order is presented. Also, the theory is valid for any nonholonomic system with nonlinear constraints of the first order.

It is well known that nonholonomic constraints produce constraint reactions. These generalized forces are, in the general case, nonconservative. In this paper a nonholonomic mechanical system of second order for any nonconservative mechanical system with the same motion as of the nonconservative system is found.

The following conventions will be observed: (1) Small italic indices unrepeated imply a range of values from 1 to n; (2) small Greek indices unrepeated imply a range of values from 1 to k, $k < n$; (3) the summation convention is employed throughout; (4) those indices, which do not imply tensorial character with respect to coordinate transformations, are enclosed in brackets. The rule is not observed in the case of the connection coefficients.

§2. Let us consider a mechanical system, whose configuration is described by n generalized coordinates q^i, and the motion is restricted by k nonholonomic constraints of the second order

$$C_{(a)i}(t, q^r, \dot{q}^r)\ddot{q}^i + C_{(a)}(t, q^r, \dot{q}^r) = 0, \quad (\cdot \equiv d/dt), \qquad ...(1)$$

where $C_{(a)i}$, $C_{(a)}$ are given functions of time t, generalized coordinates q^r and generalized velocities $\dot{q}^r$. The kinetic energy of the system is

$$T = g_{ij}\dot{q}^i \dot{q}^j/2, \qquad ...(2)$$

where g_{ij} is the fundamental covariant tensor of second order which is a function of position q^i.

Assuming that the nonholonomic constraints (1) are ideal, the governing equations of motion in contravariant form are (Djukic 1972)

$$\frac{\overset{g}{D}\dot{q}^i}{dt} \equiv \ddot{q}^i + \left\{ {i \atop k\ s} \right\} \dot{q}^k \dot{q}^s = Q^i + \mu_{(a)} C^i{}_{(a)}, \qquad ...(3)$$

where $\left\{ {i \atop k\ s} \right\}$ is Christoffel's symbol formed with respect to the tensor g_{ij}, the symbol $\overset{g}{D}/dt$ designates the absolute derivative with respect to the same tensor g_{ij}, $Q^i = Q^i(t, q^r, \dot{q}^r)$ are generalized forces and $\mu_{(a)}$ the Lagrange's multipliers. Substituting (3) into (1), we obtain

$$C_{(\nu)i}Q^i + \mu_{(a)} C^i{}_{(a)} C_{(\nu)i} - C_{(\nu)i} \left\{ {i \atop k\ s} \right\} \dot{q}^k \dot{q}^s + C_{(\nu)} = 0. \qquad ...(4)$$

(*a*) In the case of one nonholonomic constraint (1), from (4) we have the Lagrange's multiplier

$$\mu = (C^r C_r)^{-1} \left[C_i \left\{ {i \atop k\ s} \right\} \dot{q}^k \dot{q}^s - C_i Q^i - C \right]. \qquad ...(5)$$

(*b*) In the general case, when the vectors $C_{(a)i}$ are mutually orthogonal unit vectors (see Synge 1926, p. 54), i.e., when $C^i{}_{(\nu)} C_{(a)i} = \delta_{\nu a}$, where $\delta_{\nu a}$ is the Kronecker delta, the solution of (4) is

$$\mu_{(\nu)} = C_{(\nu)i} \left[\left\{ {i \atop k\ s} \right\} \dot{q}^k \dot{q}^s - Q^i \right] - C_{(\nu)}. \qquad ...(6)$$

In the following analysis we will distinguish these two cases.

§3·1. Let us consider the inertial motion ($Q^r = 0$) of a nonholonomic mechanical system with one homogeneous nonholonomic constraint (1) ($C = 0$). Using (3) and (5), we can prove that the following theorem is valid.

Theorem I—The differential equations of the inertial motion of a nonholonomic mechanical system with one homogeneous nonholonomic constraint of the second order are identical to the geodesic lines (7)

$$\frac{\overset{\Gamma}{D}\dot{q}^r}{dt} \equiv \ddot{q}^r + \Gamma^r{}_{ks}\dot{q}^k \dot{q}^s = 0, \qquad ...(7)$$

in a linear connected space L_n with connection coefficients

$$\Gamma^r_{ks} = \left\{ {r \atop k\ s} \right\} - \frac{C^r C_i}{C^m C_m} \left\{ {i \atop k\ s} \right\}, \qquad \ldots(8)$$

where $\overset{\Gamma}{D}/dt$ is the absolute derivative with respect to the connection coefficients Γ^r_{ks}. During the motion, time t plays the role of an affine parameter. The connection coefficients are symmetric, i.e., $\Gamma^r_{ks} = \Gamma^r_{sk}$. The covariant derivative, $\overset{\Gamma}{\nabla}$, of the fundamental tensor g_{ij} with respect to the connection coefficients (8) is given by

$$\overset{\Gamma}{\nabla}_s g_{ij} = \frac{C_k}{C^m C_m} \left(\left\{ {k \atop s\ i} \right\} C_j + \left\{ {k \atop s\ j} \right\} C_i \right). \qquad \ldots(9)$$

The expressions for the covariant derivative (9) and connection coefficients (8) are not covered by the classification given by Schouten (1954, pp. 132–33).

§3·2. If the generalized forces are not equal to zero, and if a nonholonomic constraint is nonhomogeneous, using (3) and (5), we have the following theorem.

Theorem II—A nonholonomic mechanical system with one nonholonomic constraint of the second order (1) moves in such a way that the differential equations of motion (3) are identical to the geodesic lines (10)

$$\frac{\overset{\bar\Gamma}{D}\dot q^r}{dt} \equiv \ddot q^r + \bar\Gamma^r_{ks} \dot q^k \dot q^s = 0, \qquad \ldots(10)$$

In a linear connected space $\bar L_n$ with connection coefficients (11).

$$\bar\Gamma^r_{ks} = \left\{ {r \atop k\ s} \right\} + \frac{1}{2T}(-\bar Q_k \delta_s^r + \bar Q_s \delta_k^r - \bar Q^r g_{ks}), \qquad \ldots(11)$$

where

$$\bar Q^r = Q^r + \frac{C^r}{C^m C_m} \left(C_i \left\{ {i \atop k\ s} \right\} \dot q^k \dot q^s - C_i Q^i - C \right). \qquad (12)$$

The connection coefficients (11) are the sum of the symmetric part

$$\bar\Gamma^r_{(ks)} = \left\{ {r \atop k\ s} \right\} - \frac{1}{2T} \bar Q^r g_{ks}, \qquad \ldots(13)$$

and the asymmetric part

$$\bar\Gamma^r_{[ks]} = S_{[k} \delta^r_{s]}, \qquad \ldots(14)$$

where the vectors S_k are

$$S_k = -\bar Q_k/T. \qquad \ldots(15)$$

Now, the covariant derivative of the fundamental tensor g_{ij} with respect to the connection coefficients (11) is given by

$$\overset{\bar\Gamma}{\nabla}_s g_{ij} = \bar Q_s g_{ij}/T. \qquad \ldots(16)$$

The expressions for covariant derivative (16) and for connection coefficients (11) to (15) determine the linear connected space $\bar{L}_n$ as a semi-metric and semi-symmetric space (see Schouten 1954, p. 126). The space $\bar{L}_n$ was used by Djukic and Vujanovic (1975) for the geometrization of motion of nonconservative mechanical systems.

§3·3. Using (3) and (6), in the case of a nonholonomic mechanical system, which is subject to more than one constraint (1), the corresponding theorems are as follows.

Theorem III—The differential equations of the inertial motion of a nonholonomic mechanical system with homogeneous ($C_{(\nu)} = 0$) constraints (1) are equivalent to the geodesic lines (7) in a linear connected space L_n with symmetric connection coefficients (17)

$$\Gamma^r_{ks} = \left\{ {r \atop k\,s} \right\} - C^r_{(\nu)} C_{(\nu)i} \left\{ {i \atop k\,s} \right\}. \qquad \text{...(17)}$$

In this case the covariant derivative of the fundamental tensor g_{ij} is given by

$$\overset{\Gamma}{\nabla}_s g_{ij} = C_{(\nu)r} \left[\left\{ {r \atop s\,i} \right\} C_{(\nu)j} + \left\{ {r \atop s\,j} \right\} C_{(\nu)i} \right]. \qquad \text{...(18)}$$

§3·4. If the nonholonomic mechanical system is subject to generalized forces Q^i and k nonholonomic constraints (1), then Theorem II is still valid for

$$\bar{Q}^r = Q^r + C^r_{(\nu)} \left[C_{(\nu)i} \left\{ {i \atop k\,s} \right\} \dot{q}^k \dot{q}^s - C_{(\nu)i} Q^i - C_{(\nu)} \right]. \qquad \text{...(19)}$$

§4. It is well known that the reactions of nonholonomic constraints are, in general, forces of the nonconservative type. Let the differential equation of motion of a holonomic mechanical system with kinetic energy (2), potential energy $\pi(q^r)$ and nonconservative generalized forces $\tilde{Q}_i(t, q^r, \dot{q}^r)$ be of the following form

$$\frac{\overset{g}{D}\dot{q}^r}{dt} = -\frac{\partial \pi}{\partial q^s} g^{sr} + \tilde{Q}^r. \qquad \text{...(20)}$$

These equations will be equivalent to the equations of motion of a nonholonomic mechanical system (3) if the nonholonomic system is subject to the same generalized forces $Q_i = -\partial\pi/\partial q^i$ and

$$\mu_{(\alpha)} C^r_{(\alpha)} = \tilde{Q}^r. \qquad \text{...(21)}$$

(*a*) Combining (21) and (4), in the case of one nonholonomic constraint, we have

$$C = \tilde{Q}_i \left[\left\{ {i \atop k\,s} \right\} \dot{q}^k \dot{q}^s - Q^i - \tilde{Q}^i \right] \Big/ \mu. \qquad \text{...(22)}$$

Substituting (21) and (22) into (1) we obtain

$$\tilde{Q}_i \overset{g}{D}\dot{q}^i/dt - Q^i - \tilde{Q}^i) = 0, \qquad \text{...(23)}$$

and the following theorem.

Theorem IV—The motion of a nonconservative mechanical system with nonconservative generalized forces $\tilde{Q}_r(t, q^i, \dot{q}^i)$ and potential forces $Q_i = -\partial\pi/\partial q^i$ is equivalent to the motion of a nonholonomic mechanical system with the same kinetic and potential energies, without nonconservative forces and in presence of a nonholonomic constraint of the second order (23).

(*b*) A similar consideration in the case of more than one nonholonomic constraint shows that this correspondence between the nonconservative and nonholonomic mechanical systems is not unique.

§5. Let (Lanchester's problem—problem in combat tactics, see Housner and Hudson 1959)

$$\ddot{q}^1 = -b\dot{q}^2, \quad \ddot{q}^2 = -a\dot{q}^1, \qquad \ldots(24)$$

(where a and b are constants) be the differential equations of motion of a mechanical system with two degrees of freedom, kinetic energy $T = [(\dot{q}^1)^2 + (q^2)^2]/2$ and nonconservative forces $Q_1 = -b\dot{q}^2$, $Q_2 = -a\dot{q}^1$. Applying Theorem IV it follows that this motion is equivalent to the motion of a nonholonomic system with the same kinetic energy, without nonconservative forces, and in presence of the nonholonomic constraint

$$a\dot{q}^1\ddot{q}^2 + b\dot{q}^2\ddot{q}^1 + a^2(\dot{q}^1)^2 + b^2(\dot{q}^2)^2 = 0. \qquad \ldots(25)$$

Acknowledgement

The author wishes to acknowledge the assistance of Dr. A. Strauss for his constructive review of the paper.

References

Djukic, Dj. S. (1972). The canonical equations of mechanical systems with the nonholonomic constraints of the second order. *Mat. Vesnik*, **9**, 351–62.

Djukic, Dj. S., and Vujanovic, B. (1975). On some geometrical aspects of classical nonconservative mechanics. *J. Math. Phys.*, **16**, 2099–2102.

Dobronravov, V. V. (1961). Control systems as the systems with nonholonomic constraints. *Mechanics, Sbornik statei* 104, *Oboron-giz, Moscow*, pp. 27–32.

Douglas, J. (1928). The general geometry of path. *Ann. Math.*, **2**, 29.

Housner, G. W. and Hudson, D. E. (1959). Applied Mechanics—Dynamics, second edition, D. Van Nostrand Co., Princeton, p. 73.

Schouten, J. A. (1954). Ricci-Calculus. Springer-Verlag, Berlin.

Synge, J. L. (1926). On the geometry of dynamics. *Phil. Trans. R. Soc.*, A **226**, 31–106.