

M-HYPONORMAL OPERATORS : INVARIANT SUBSPACES AND SPECTRAL LOCALIZATION

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Additional properties for a new class of operators called M -hyponormal operators are obtained. Invariant subspaces, reducing subspaces of such operators are discussed. Local spectralization and analytic extensions for M -hyponormal operators are studied.

1. INTRODUCTION

The study of hyponormal operators on a Hilbert space has of late become quite extensive. Stampfli (unpublished) has initiated the study of a new class of operators known as M -hyponormal operators. Wadhwa (1974) investigated some spectral theoretic aspects of M -hyponormal operators in his doctoral dissertation submitted to the Indiana University.

The purpose of this paper is to obtain additional information about this class of operators. In the course of investigations we discuss the invariant subspaces reducing subspaces of such operators. Further we show that on every finite dimensional Hilbert space M -hyponormality collapses to normality. After generalizing the concept of quadratically hyponormal due to Stampfli (1966) to the case of M -hyponormal operator we investigate the local spectral theory of these operators. In particular we show that a quadratically M -hyponormal operator is, under some conditions, similar to a normal operator.

A more complete discussion of the results is delayed (largely because of their disconnected nature) until the results themselves.

We begin with introducing some notations, definitions which are needed in the sequel. We shall always use H to denote a complex Hilbert space. For an operator T on H , $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_R(T)$, $\sigma_c(T)$, $\rho(T)$ will stand for the spectrum approximate point spectrum, the residual spectrum, the continuous spectrum and the resolvent set of T respectively. For $x \in H$, $(T - zI)^{-1}R(T, x, z)$ is an analytic vector valued function for z in $\rho(T)$. A vector-valued function $f(z)$ is called an

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analytic extension of $R(T, x, z)$ if it is defined and analytic on an open set $D(f)$ containing $\rho(T)$ and if $(T - zI)f(z) = x$ for z in $D(f)$. A maximal single valued extension $\text{Re}(T, x, z)$ is the union of all extensions of $R(T, x, z)$. Further we set $\rho(T, x) = \{z: \text{Re}(T, x, z) \text{ is analytic at } z\}$ and, $\sigma(T, x) = \text{complement of } \rho(T, x)$. All these concepts are due to Dunford. In the following \mathbb{C} denotes the complex field.

2. GENERAL PROPERTIES

In this section we give the definition of M -hyponormal operator and we shall state some general properties of it.

Definition—An operator T on H is said to be an M -hyponormal operator if there exists a real number M such that

$$\|(T - zI)^* x\| \leq M \|(T - zI)x\| \quad \dots(1)$$

for all x in H and for every complex z .

We also note the observations:

(i) If T is an M -hyponormal operator, then $M \geq 1$ and T is hyponormal if and only if $M = 1$.

(ii) If T is an M -hyponormal operator, then $\|(T^* - \bar{w}I)^{-1}x\| \leq M \|(T - wI)^{-1}x\|$ for all w in the resolvent set of T (see Wadhwa 1974a, Prop. 2), i.e., $(T - wI)^{-1}$ is an operator for which (1) holds at $z = 0$, and for all w in the resolvent set of T .

We enlist here some results the proofs of which are straightforward.

Theorem 2.1—If T is an M -hyponormal operator on H , then λT , ($\lambda \neq 0$) and $(T + \lambda I)$ are also M -hyponormal operators for every scalar λ .

Theorem 2.2—Let T be unitary operator on H , P be a positive and invertible operator on H , $C = P^{-1}UP$ and $T = UP$, then $\|C\| \leq M$ if T is M -hyponormal.

Theorem 2.3—If T is an M -hyponormal operator, then $\text{Re } \sigma_{ap}(T) \subset \sigma(\text{Re } T)$ (see Lemma 1 of Berberian 1971 for hyponormal case).

Theorem 2.4—Let $\{T_n\}$ be a sequence of M -hyponormal operators converging uniformly to the M -hyponormal operator T , so that $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$; then z_0 is in $\sigma(T)$ if and only if there exists a sequence $\{z_n\}$, z_n are in $\sigma(T_n)$ such that $z_n \rightarrow z_0$. [The method of the proof is essentially similar to that given by Clancey and Putnam (1972)].

Theorem 2.5—If T is an M -hyponormal operator, then $\sigma_{ap}(T^*) = \sigma(T^*)$.

Theorem 2.6—Let T be an M -hyponormal operator and let z_1, z_2 be in $\sigma_{ap}(T)$ with $z_1 \neq z_2$. If $\{x_n\}$ and $\{y_n\}$ are the sequences of unit vectors of H such that $\|(T - z_1I)x_n\| \rightarrow 0$ and $\|(T - z_2I)y_n\| \rightarrow 0$ then $(x_n, y_n) \rightarrow 0$.

Theorem 2.7—Let T be an M -hyponormal operator and P be a self-adjoint operator on H . If TP is a contraction, then $\|PT\| \leq M$.

PROOF: By M -hyponormality of T , we have

$$\|T^*Px\| \leq M \|TPx\| \leq M \|x\|, \text{ (for } \|TP\| \leq 1).$$

Hence $\|PT\| = \|(PT)^*\| = \|T^*P\| \leq M$.

Remark—If in *Theorem 2.7*, T is hyponormal, then PT is a contraction.

Saito (1975) has presented the factorization of the hyponormal operators on a Hilbert space. Here we take up similar considerations for the M -hyponormal operators.

Theorem 2.8—If T is M -hyponormal operator on H , then there exist operators A, B and S which satisfy

- (i) $B \geq A \geq 0$,
- (ii) $\|S\| \leq M, M \geq 1$,
- (iii) $S^*AS = B$.

In this case, T can be expressed as

$$T = \frac{1}{M}(A^{1/2}S + \lambda I)$$

for some complex λ .

PROOF: Let T be M -hyponormal operator. Put $T - zI = V$ where z is any complex number. Let $V^* = U(VV^*)^{1/2}$ be a polar decomposition of V^* . Put $A = VV^*$ and $B = M^2V^*V$. Then, since T is M -hyponormal, we have $B \geq A \geq 0$. Also we have, $B = M^2V^*V = M^2U(VV^*)^{1/2}(VV^*)^{1/2}U^* = M^2UVV^*U^* = M^2UAU^*$. Let $S = MU^*$. Then $\|S\| \leq M$ and $B = S^*AS$. Now $T = V + zI = (VV^*)^{1/2}U^* + zI = (1/M)A^{1/2}S + zI = 1/M(A^{1/2}S + \lambda I)$ where λ is some complex number. Hence the proof is complete.

Yoshino (1967) has established the spectral resolution of hyponormal operators under certain condition which we observe to be valid for M -hyponormal operators in the following theorem. The proof also follows on the lines of Yoshino (1967).

Theorem 2.9—If T is M -hyponormal operator on H with $\sigma(T) = \sigma_c(T)$ lying on C^1 -Jordan curve and if T satisfies growth condition, i.e.

$$\|(T - zI)^{-1}\| \leq \{d(z, \sigma(T))\}^{-1}, \forall z \in \rho(T);$$

then

$$H = H(s_\alpha, s_\beta) \oplus H(s_\beta, s_\alpha + l(c))$$

and

$$H(s_\alpha, s_\beta), H(s_\beta, s_\alpha + l(c)) \text{ reduce } T;$$

where

$$H(s_\alpha, s_\beta) = \{x \in H: \sigma(T, x) \subset \text{arc}(g(s_\alpha), g(s_\beta))\}$$

and

$$H^*(s_\alpha, s_\beta) = \{x \in H: \overline{\sigma(T^*, x)} \subset \text{arc}(g(s_\alpha), g(s_\beta))\}.$$

It is to be noted that the notations in Theorem 2.9 are essentially same to those of Yoshino (1967).

3. INVARIANT SUBSPACES

In this section we shall deal with the closed subspaces of H in connection with an M -hyponormal operator.

Theorem 3.1—If T is an M -hyponormal operator on H , then the set

$$K = \{x \in H: \|(T^* - \bar{z}I)x\| = M \|(T - zI)x\|, z \in \mathbb{C}\}$$

is a closed subspace of H .

PROOF: For $x \in K$, we have,

$$\|(T^* - \bar{z}I)x\|^2 = M^2 \|(T - zI)x\|^2,$$

which yields

$$\{(M^2(T^* - \bar{z}I)(T - zI) - (T - zI)(T^* - \bar{z}I))x, x\} = 0. \tag{2}$$

In view of M -hyponormality of T , (2) holds if and only if

$$\{M^2(T^* - \bar{z}I)(T - zI) - (T - zI)(T^* - \bar{z}I)\}x = 0. \tag{3}$$

From (3) it follows that K is the null space of the operator

$$M^2(T^* - \bar{z}I)(T - zI) - (T - zI)(T^* - \bar{z}I).$$

Hence K is a closed subspace as desired.

Remarks—(1) If T^* is M -hyponormal operator on H , then the set $K = \{x \in H: \|(T - zI)x\| = M \|(T^* - \bar{z}I)x\|, z \in \mathbb{C}\}$ is a closed subspace of H .

(2) For hyponormal operator T (i.e., when $M = 1$) it is well known that the space K is invariant under T , and T/K is normal.

(3) It will be interesting to know whether an M -hyponormal operator has a proper invariant subspace; as K constructed above does not serve our purpose.

4. M-HYPONORMALITY IMPLYING NORMALITY

In this section we shall be concerned with the eigenspaces of M -hyponormal operators. The motivation of the results is due to Halmos (1967, Problem 163) and Berberian (1961).

Theorem 4.1—Suppose that the subspace K of H reduces an operator T on H . Then T is M -hyponormal if and only if T/K and T/K^\perp are M -hyponormal.

PROOF: Denote $T/K = T_1$ and $T/K^\perp = T_2$. If T is M -hyponormal, then there exists a real number M such that $\| (T - zI)^* x \| \leq M \| (T - zI) x \|$ for all x in H and for every complex z . Note that $T = T_1$ and $T^* = T_1^*$ on K . For a vector x in K we have,

$$\begin{aligned} \| (T_1 - zI)^* x \| &= \| (T - zI)^* x \| \leq M \| (T - zI) x \| \\ &= M \| (T_1 - zI) x \|. \end{aligned}$$

This shows that T_1 is M -hyponormal.

Again for x in K^\perp we have,

$$\begin{aligned} \| (T_2 - zI)^* x \| &= \| (T - zI)^* x \| \leq M \| (T - zI) x \| \\ &= M \| (T_2 - zI) x \| \end{aligned}$$

showing that T_2 is M -hyponormal.

Conversely assume that T_1 and T_2 are M -hyponormal operators. We know that every x in H can be written as $x = x_1 + x_2$ where x_1 is in K , x_2 is in K^\perp . Hence for all complex z and for all vectors x in H we have,

$$\begin{aligned} \| (T - zI)^* x \|^2 &= \| (T - zI)^* x_1 + (T - zI)^* x_2 \|^2 \\ &= \| (T_1 - zI)^* x_1 + (T_2 - zI)^* x_2 \|^2 \\ &= \| (T_1 - zI)^* x_1 \|^2 + \| (T_2 - zI)^* x_2 \|^2 \\ &\leq M^2 \| (T_1 - zI) x_1 \|^2 + M^2 \| (T_2 - zI) x_2 \|^2 \\ &= M^2 \| (T - zI) x_1 \|^2 + M^2 \| (T - zI) x_2 \|^2 \\ &= M^2 \| (T - zI) x \|^2 \end{aligned}$$

by which M -hyponormality of T is obvious.

Theorem 4.2—Let T be an M -hyponormal operator. Then the span of all eigenvectors of T reduces T .

PROOF: We shall divide the proof in four steps.

Step 1—Firstly,

$$\{x \in H: Tx = \lambda x\} \subset \{x \in H: T^*x = \bar{\lambda}x\}$$

for all complex λ and this fact is obvious.

Step 2—For each complex λ , the subspace $\{x \in H : Tx = \lambda x\}$ reduces T .

Denote by K the subspace $\{x \in H : Tx = \lambda x\}$. For x in K , we have $T(Tx) = \lambda(Tx)$ which implies that Tx is in K . Also $T(T^*x) = \bar{\lambda}(Tx) = \lambda(\bar{\lambda}x) = \lambda(T^*x)$ showing that T^*x belongs to K . Thus K reduces T .

Step 3—If $\lambda_1 \neq \lambda_2$, then $\{x \in H : Tx = \lambda_1 x\} \perp \{x \in H : Tx = \lambda_2 x\}$. The proof of this fact is simple [see Wadhwa 1974a, Prop. 2 (i)].

Step 4—The span of all the eigenvectors of T reduces T and the restriction of T to that span is normal.

The proof follows from steps (1), (2), (3) and using the fact that the restriction of T to any of the eigenspaces of itself is normal; which of course follows from step (2).

Theorem 4.3—If H is finite-dimensional and T is an M -hyponormal operator on H , then T is normal.

PROOF: We shall follow the Induction Method. For $n = 1$, every operator is normal. Assume that the theorem holds for dimension $< n$. It can be shown that every linear mapping in a finite dimensional space has at least one eigenvalue. Let μ be an eigenvalue for T . Let $K = K_T(\mu)$ be the eigenspace of T associated with μ . By Theorem 4.2, K reduces T and T/K is normal. It can be seen that K^\perp also reduces T . This fact along with Theorem 4.1 leads us to the conclusion that T/K^\perp is M -hyponormal. Hence T/K^\perp is normal by the inductive assumption and we are through.

Remarks: The method of proof is analogous to that given in Berberian (1961, p. 198).

5. QUADRATICALLY M -HYPONORMAL OPERATORS

Stampfli (1966) has discussed quadratically hyponormal operator on H . Here in this section we shall generalize this concept for M -hyponormal operators. We shall also extend the concept of local spectral theory, which was discussed by Stampfli (1966, Theorem 3), to M -hyponormal operator.

Definition—We say that an operator T on H is quadratically M -hyponormal if $aT^2 + bT + c$ is M -hyponormal for all complex a, b, c .

Remarks—(1) For $M = 1$, quadratically M -hyponormal operator will be precisely quadratically hyponormal.

(2) An M -hyponormal operator is not necessarily quadratically M -hyponormal.

Lemma 5.1—If T is a quadratically M -hyponormal operator, $(aT^2 + bT + c)^{-1}$ exists for all complex a, b, c ; then we have

$$\| \{(aT^2 + bt + c)^*\}^{-1} x \| \leq M \| (aT^2 + bT + c)^{-1} x \|.$$

PROOF: Proof follows from the definition of quadratically M -hyponormality and observation (ii) in section 2.

Definition—A hyponormal operator T on H is called completely hyponormal if there is no reducing subspace of T on which T is normal.

Theorem 5.1—Let T be a completely hyponormal and quadratically M -hyponormal operator such that $\sigma_R(T) = \phi$. Then $\overline{\sigma(T^*, x)} \subset \sigma(T, x)$ for all x in H .

PROOF: Since T is completely hyponormal with $\sigma_R(T) = \phi$, the function $f: \overline{\rho(T, x)} \rightarrow H$ defined by $f(z) = (T^* - zI)^{-1}x$ for $\bar{z} \in \rho(T, x)$ is continuous in the weak topology for fixed x in H and analytic for \bar{z} in $\rho(T)$ (see Stampfli 1966, Theorem 1, p. 290).

Now let \bar{z} be in $\rho(T, x)$; since $\text{Re}(T, x, z)$ is analytic for z in $\rho(T, x)$; $\{(T - zI)^{-1} - (T - \bar{z}_0I)^{-1}\}x = (z - \bar{z}_0)(T - zI)^{-1}(T - \bar{z}_0I)^{-1}x$ is bounded for z in a suitably chosen closed neighbourhood of \bar{z}_0 . Putting this in another way $\|(T - zI)^{-1}(T - \bar{z}_0I)^{-1}x\| \leq M$ for $|z - \bar{z}_0| \leq \delta$, for some $\delta < 0$, this is true for $z = \bar{z}_0$ in particular and hence by using Lemma 5.1, we have

$$\begin{aligned} \|(T^* - \bar{z}_0I)^{-2}x\| &= \|(T^* - 2\bar{z}_0T^* + \bar{z}_0^2I)^{-1}x\| = \|\{(T^2 - 2z_0T + z_0^2I)^*\}x\| \leq M \|(T^2 - 2z_0T + z^2I)^{-1}x\| = M \|(T - z_0I)^{-2}x\| \\ &\leq MK \text{ for } |z - z_0| \leq \delta. \end{aligned}$$

Again,

$$\begin{aligned} \|(T^* - zI)^{-1}(T^* - z_0I)^{-1}x\| &= \|\{(T^* - z_0I)(T^* - zI)\}^{-1}x\| \\ &\leq M \|\{T^2 - (\bar{z}_0 + \bar{z})T + \bar{z}\bar{z}_0I\}^{-1}x\| = M \|\{(T - \bar{z}_0I) \\ &\quad \times (T - \bar{z}I)\}^{-1}x\| = M \|(T - \bar{z}I)^{-1}(T - \bar{z}_0I)^{-1}x\| \\ &\leq MK \text{ for } |z - z_0| \leq \delta. \end{aligned}$$

Now we shall show that $f(z)$ has a derivative at z_0 . This is equivalent to showing that

$$\lim_{z \rightarrow z_0} \left(\frac{[(T^* - zI)^{-1} - (T^* - z_0I)^{-1}]x, y}{z - z_0} \right) = ((T^* - z_0I)^{-2}x, y)$$

which in turn is equivalent to the statement

$$\lim_{z \rightarrow z_0} ((T^* - zI)^{-1}(T^* - z_0I)x, y) = ((T^* - z_0I)^{-2}x, y); \text{ and } y$$

is in H . Given y , choose a vector v in H such that $\|(T - \bar{z}_0I)v - y\| < \varepsilon$, and then we have

$$\begin{aligned} &| \{[(T^* - zI)^{-1} - (T^* - z_0I)^{-1}](T^* - z_0I)^{-1}x, y\} | \\ &\leq 2MK\varepsilon + | \{[(T^* - zI)^{-1} - (T^* - z_0I)^{-1}](T^* - z_0I)^{-1}x, (T - \bar{z}_0I)v\} | \\ &= 2MK\varepsilon + | \{[(T^* - zI)^{-1} - (T^* - z_0I)^{-1}](T^* - z_0I)^{-1}x, v\} | \end{aligned}$$

$$\begin{aligned}
 &= 2MK\varepsilon + |z - z_0| \left| \left((T^* - zI)^{-1} (T^* - z_0I)^{-1} x, v \right) \right| \\
 &\leq 2MK\varepsilon + |z - z_0| MK \|v\| \text{ for } |z - z_0| \text{ sufficiently small.}
 \end{aligned}$$

Thus $f'(z_0)$ exists. Hence by a well-known theorem (see Hille 1959) $(T^* - z_0I)^{-1}x$ may be continued analytically across the arc $\overline{\sigma(T) \cap \rho(T, x)}$ which implies that $\overline{\rho(T^*, x)} \supseteq \overline{\rho(T, x)}$.

Thus we get the required result.

Remark—It is observed that the method of proof of this result is due to Stampfli (1966, Theorem 3).

Now from Theorem 4.6 of Stampfli (1966) we have the following:

Corollary 1—Let T be a completely hyponormal and quadratically M -hyponormal operator on H such that $\sigma_R(T) = \phi$. Then $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies that $(x, y) = 0$.

In particular Corollary 1 implies that T satisfies Dunford's boundedness condition (B), namely

$$\sigma(T, x) \cap \sigma(T, y) = \phi \text{ implies } \|x\| \leq K \|x + y\|,$$

where K is independent of x, y and depends only on T .

Theorem 5.2—Let $\sigma(T) = A \cup B$ where A and B are closed, connected and disjoint, and let T be quadratically M -hyponormal, and completely hyponormal operator on H with $\sigma_R(T) = \phi$. Then $H = H_1 \oplus H_2$, where H is a reducing subspace of T ; $\sigma(T/H_1) = A$ and $\sigma(T/H_2) = B$.

PROOF: By similar arguments as in Theorem 5 of Stampfli (1966) and using Theorem 5.1, one can establish the proof.

Theorem 5.3—Let T be a completely hyponormal and quadratically M -hyponormal operator on H such that $\sigma_R(T) = \phi$. Let $\sigma(T)$ lie on a C^1 -Jordan curve and suppose that T satisfies the growth condition (G_1) , that is

$$\|(\lambda I - T)^{-1}\| \leq \frac{K}{\text{dist.}[\lambda, \sigma(T)]} \text{ for all } \lambda \text{ in } \rho(T),$$

$|\lambda| \leq \|T\| + 1$. Then T is similar to a normal operator.

PROOF: Since T satisfies the requirements of Corollary 5.1, it follows that T satisfies Dunford's boundedness condition (B), and since T satisfies the growth condition (G_1) , therefore by Theorem 14 and Theorem 7 of Wadhwa (1974) T is similar to a normal operator.

6. M -HYPONORMAL ELEMENTS

The concept of M -hyponormal elements in $B(H)$ can easily be extended to any general C^* -algebra. In this section we obtain some results for M -hyponormal elements in a C^* -algebra and Calkin algebra $a(H)$ (the algebra of all operators modulo the ideal of compact operators on H).

Theorem 6.1—Let \mathcal{G} be a C^* -algebra with unit and let A be an M -hyponormal element of \mathcal{G} . Then for any P in \mathcal{G} , $(A - z)P = 0$ implies that $(A - z)^*P = 0$, where z is any complex number.

PROOF: Since A is M -hyponormal, we have

$$(A - z)(A - z)^* \leq M^2(A - z)^*(A - z).$$

For any P in \mathcal{G} ,

$$P^*(A - z)(A - z)^*P \leq M^2P^*(A - z)^*(A - z)P.$$

If $(A - z)P = 0$, then $P^*(A - z)(A - z)^*P = 0$.

This implies that $(A - z)^*P = 0$.

Theorem 6.2—Let A be an M -hyponormal element of a C^* -algebra with unit. If $AP_i = \lambda_i P_i$, $i = 1, 2$ and if $\lambda_1 \neq \lambda_2$, then $P_1 P_2 = 0$.

PROOF: Let $AP_1 = \lambda_1 P_1$ and $AP_2 = \lambda_2 P_2$.

Then by Theorem 6.1, $A^*P_1 = \bar{\lambda}_1 P_1$ and $A^*P_2 = \bar{\lambda}_2 P_2$ which implies that $P_1 A_1 = \lambda_1 P_1$ and $P_2 A = \lambda_2 P_2$.

Now

$$\lambda_2 P_1 P_2 = P_1 (\lambda_2 P_2) = P_1 A P_2 = (P_1 A) P_2 = \lambda_1 P_1 P_2.$$

Thus $P_1 P_2 = 0$, since $\lambda_1 \neq \lambda_2$.

Remark—Result analogous to Theorem 6.1 and Theorem 6.2 for hyponormal elements have been proved by Fillmore *et al.* (1972). Salinas (1973) has proved that for hyponormal \hat{T} [Image of $T \in B(H)$ in the Calkin algebra $a(H)$] $R_e(T) = E_1(T)$ where $R_e(T)$ is the set of all reducing essential eigenvalues of T and $E_1(T)$ is the left essential spectrum of T (see Salinas 1973, for notations). We extend this result to M -hyponormal element in $a(H)$ in the following theorem, the proof of which is verbatim to that of Salinas (1973).

Theorem 6.3—Let T be an operator on H and suppose that \hat{T} is M -hyponormal in $a(H)$. Then $R_e(T) = E_1(T)$.

Remark—Theorem 6.3 is in fact another version of Theorem 6.1.

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Addendum: The author has recently improved Theorem 5.1 to completely M -hyponormal operators.