ON HYPERSURFACES OF RECURRENT FINSLER SPACES

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In the present paper, Gauss characteristic equations of a umbilical hypersurface immersed in a Finsler space have been deduced from the standpoint of the Berwald connection. The conditions under which a hypersurface immersed in a recurrent Finsler space is recurrent, have been investigated. Several properties of the hypersurface have been derived for the following two cases: (i) The recurrence vector field of the embedding space is not normal to the hypersurface, and (ii) the recurrence vector field is normal to the hypersurface.

INTRODUCTION

Miyazawa and Chūman (1972) have studied umbilical subspaces of recurrent Riemannian spaces. Singh and Singh (1976) have investigated the properties of umbilical subspaces immersed in recurrent Finsler space. This study is based on the Cartan's process of covariant differentiation. In this paper, we shall study the properties of umbilical hypersurfaces immersed in a Finsler space which is recurrent in the sense of Berwald.

1. BASIC CONCEPTS

Consider an n-dimensional Finsler space F_n referred to a local coordinate system x^i, whose metric function $F(x, \dot{x})$ satisfies the condition usually imposed on a Finsler metric (Rund 1959, p. 16). The metric tensor defined by $g_{ij}(x, \dot{x}) = \frac{1}{2}\, \dot{\partial}^2_{ij} F^2(x, \dot{x})$ is positively homogeneous of degree zero in $\dot{x}^i$ where $\dot{\partial}^2_{ij}$ stands for $\partial^2/\partial \dot{x}^i\, \partial \dot{x}^j$.

The Berwald's and Cartan's connection coefficients G^i_{jk} and $\overset{*}{\Gamma}{}^i_{jk}$ are related by

$$G^i_{jk} = \overset{*}{\Gamma}{}^i_{jk} + C^i_{jk|r}\dot{x}^r, \qquad \text{...(1.1)}$$

where

$$C^i_{jk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial \dot{x}^k},$$

is a symmetric tensor and the symbol '$|$' stands for the Cartan's process of covariant differentiation.

Consider a hypersurface F_{n-1} given by the equations $x^i = x^i(u^\alpha)$, $(i = 1, \ldots\ldots, n;\ \alpha = 1, \ldots\ldots, n-1)$. The components $g_{ij}(x, \dot{x})$, $g_{\alpha\beta}(u, \dot{u})$ of the metric tensor of F_n and F_{n-1} are related by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x})\, B^i_\alpha B^j_\beta, \qquad \ldots(1.2)$$

where

$$B^i{}_\alpha = \frac{\partial x^i}{\partial u^\alpha}.$$

The unit normal vector N_i satisfies the relations

$$(a)\quad N_i B^i{}_\alpha = 0, \quad (b)\ g_{ij} N^i N^j = 1 \qquad \ldots(1.3)$$

where

$$N^i = g^{ij} N_j.$$

The Cartan's induced and intrinsic connection coefficients $\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ and $'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ (Rund 1959, p. 213) are related by

$$'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma} = \Lambda^\alpha_{\beta\gamma} + \overset{*}{\Gamma}{}^\alpha_{\beta\gamma} \text{ (Rund 1965)} \qquad \ldots(1.4)$$

where

$$g_{\epsilon\gamma}\Lambda^\epsilon_{\alpha\beta} = \Lambda_{\alpha\gamma\beta} = (M_{\beta\gamma}\Omega_{\alpha\sigma} + M_{\alpha\gamma}\Omega_{\beta\sigma} - M_{\alpha\beta}\Omega_{\gamma\sigma})\,\dot{u}^\sigma$$
$$- (M_{\lambda\alpha}C^\lambda_{\beta\gamma} + M_{\lambda\beta}C^\lambda_{\alpha\gamma} - M_{\lambda\gamma}C^\lambda_{\beta\alpha})\,\Omega_{\sigma\mu}\dot{u}^\sigma \dot{u}^\mu,$$

$$M_{\alpha\beta} = M_{ij}B^{ij}_{\alpha\beta} \text{ and } M_{ij} = C_{ijk}N^k. \qquad \ldots(1.5)$$

Here

$$k_n(u, \dot{u}) = (\Omega_{\sigma\lambda}\dot{u}^\sigma \dot{u}^\lambda)\, F^{-2}(u, \dot{u}) \qquad \ldots(1.6)$$

represents the normal curvature of F_{n-1} in the direction of $\dot{u}^\sigma$. Normalizing the condition $F^2(u, \dot{u}) = 1$, equation (1.6) gives

$$k_n(u, \dot{u}) = \Omega_{\sigma\lambda}\dot{u}^\sigma u^\lambda. \qquad \ldots(1.7)$$

Therefore, the quantities $\Lambda^\epsilon_{\alpha\beta}$ satisfy the following identities :

$$(a)\quad \Lambda^\epsilon_{\alpha\beta}\,\dot{u}^\alpha = k_n M_\beta{}^\epsilon,$$

$$(b)\quad g_{\epsilon\gamma}\dot{u}^\gamma \Lambda^\epsilon_{\alpha\beta} = -k_n M_{\alpha\beta},$$

where

$$M_\beta{}^\epsilon = g^{\epsilon\gamma} M_{\gamma\beta}. \qquad \ldots(1.8)$$

The mixed covariant derivative of an arbitrary vector field T^i_α is given by

$$T^i_{\alpha(\gamma)} = \partial_\gamma T^i_\alpha - \dot{\partial}_\epsilon T^i_\alpha\, \dot{\partial}_\gamma G^\epsilon - T^i_\epsilon G^\epsilon_{\alpha\gamma} + T^r_\alpha G^i_{rh} B^h_\gamma,$$

where $G^{\epsilon}_{\alpha\gamma}$ is the induced connection parameter in the sense of Berwald and $\dot{\partial}_i$ and ∂_i represent the partial derivatives with respect to $\dot{x}^i$ and x^i respectively. In particular,

$$B^i_{\alpha\,(\beta)} = V^i_{\alpha\beta} = B^i_{\alpha\beta} - B^i_{\epsilon}\, G^{\epsilon}_{\alpha\beta} + G^i_{hk}\, B^{hk}_{\alpha\beta}\,. \qquad \ldots(1\cdot 9)$$

This can also be expressed as (Sinha and Singh 1971)

$$V^i_{\alpha\beta} = N^i\,\Omega_{\alpha\beta} - B^i_{\epsilon}\,(\Lambda^{\epsilon}_{\alpha\beta} + C^{\epsilon}{}_{\alpha\beta|\sigma}\,\dot{u}^{\sigma}) + C^i{}_{hk|\gamma}\,\dot{x}^{\gamma}\, B^{hk}_{\alpha\beta}, \qquad \ldots(1\cdot 10)$$

where $\Omega_{\alpha\beta}$ are the components of the second fundamental tensor. The following equation of Gauss for Berwald's curvature tensor has been obtained by Sinha and Singh (1971):

$$\begin{aligned} H_{\epsilon\delta\beta\gamma} = {} & H_{hlkj}B^{hlkj}_{\epsilon\delta\beta\gamma} + (\Omega_{\epsilon\beta}\Omega_{\delta\gamma} - \Omega_{\epsilon\gamma}\Omega_{\delta\beta}) + 2M_{lh}B^h{}_{\delta}\,(\Omega_{\epsilon\beta}V^l{}_{\gamma\sigma} - \Omega_{\epsilon\gamma}V^l{}_{\beta\sigma})\,\dot{u}^{\sigma} \\ & - 2N^hC_{hlk|r}\dot{x}^r\,\Omega_{\epsilon\,[\beta}B^{kl}_{\gamma]\delta} + 2B^l{}_{\delta}\,g_{il}\Lambda^{\sigma}{}_{\epsilon\,[\beta}V^i{}_{\gamma\,]\sigma}{}^{\dagger} \\ & + 2\Lambda^{\alpha}{}_{\epsilon[\beta(\gamma)]}\,g_{\alpha\delta} - 2C_{hlk|r(j)}\,\dot{x}^r\,B^{hlkj}_{\epsilon\delta\,[\beta\gamma]} - 2\dot{\partial}_jC_{hlk|r}\dot{x}^r\,B^{hlk}_{\epsilon\delta\,[\beta}V^j{}_{\gamma]\,\sigma}\dot{u}^{\sigma} \\ & - 2C_{hlk|r}B^{hlk}_{\epsilon\delta\,[\beta}V^r{}_{\gamma]\sigma}\dot{u}^{\sigma} - 2C_{hlk|r}\dot{x}^r\,B^{lk}_{\delta\,[\beta}V^h{}_{\gamma]\epsilon}\,, \end{aligned} \qquad \ldots(1\cdot 11)$$

where H_{hlkj} and $H_{\epsilon\delta\beta\gamma}$ are Berwald's curvature tensors in F_n and F_{n-1}.

2. Umbilical Hypersurfaces

Let the hypersurface F_{n-1} be umbilical, i.e.,

$$\Omega_{\epsilon\beta}\,(u, \dot{u}) = k_n g_{\epsilon\beta}\,(\dot{u}, u). \qquad \ldots(2\cdot 1)$$

The mean curvature vector $\overset{*}{M}{}^i$ of a hypersurface is given by

$$\overset{*}{M}{}^i = \frac{g^{\alpha\beta}\,\Omega_{\alpha\beta}N^i}{n-1}\,. \qquad \ldots(2\cdot 2)$$

Equations (2·1) and (2·2) give

$$g_{ij}\overset{*}{M}{}^i\,\overset{*}{M}{}^j = \overset{*}{M}_j\overset{*}{M}{}^j = k_n^{\,2}. \qquad \ldots(2\cdot 3)$$

In view of eqns. (2·1), (2·3) and the facts

$$M_{lh}N^l\,B^h{}_{\delta} = M_{\delta},\ \dot{x}^k{}_{(r)} = 0,\ C^i{}_{hk}\dot{x}^k = 0,$$

$$u^{\alpha}{}_{|\sigma} = 0 \text{ and } C^{\epsilon}_{\alpha\beta}\dot{u}^{\alpha} = 0$$

the equation of Gauss given by (1·11) will take the form

$$\begin{aligned} H_{\epsilon\delta\beta\gamma} = {} & H_{hlkj}B^{hlkj}_{\epsilon\delta\beta\gamma} + (\overset{*}{M}{}^i\,\overset{*}{M}_i)\,(g_{\epsilon\beta}g_{\delta\gamma} - g_{\epsilon\gamma}g_{\delta\beta}) \\ & + 2M_{\delta}k_n^{\,2}\,(g_{\epsilon\beta}g_{\sigma\gamma} - g_{\epsilon\gamma}g_{\sigma\beta})\,\dot{u}^{\sigma} + P_{\epsilon\delta\beta\gamma}, \end{aligned} \qquad \ldots(2\cdot 4)$$

† $2x_{[\alpha\beta]} = x_{\alpha\beta} - x_{\beta\alpha}$.

where

$$P_{\epsilon\delta\beta\gamma} = 2M_{lh}B^{hl}_{\delta\lambda}(\Lambda^{\lambda}_{\beta\sigma}\Omega_{\epsilon\gamma} - \Lambda^{\lambda}_{\gamma\sigma}\Omega_{\epsilon\beta})\,\dot{u}^{\sigma}$$
$$- 2N^{h}C_{hlk|r}\dot{x}^{r}\,\Omega_{\epsilon[\beta}B^{kl}_{\gamma]\delta} + 2B_{\delta}{}^{l}g_{il}\Lambda^{\sigma}{}_{\epsilon[\beta}V^{i}{}_{\gamma]\sigma}$$
$$+ 2\Lambda^{a}{}_{\epsilon\,[\beta\,(\gamma)]}\,g_{a\delta} - 2C_{hlk|r\,(j)}\,\dot{x}^{r}\,B^{hlkj}_{\epsilon\delta\,[\beta\gamma]}$$
$$- 2\dot{\partial}_{j}C_{hlk|r}\dot{x}^{r}\,B^{hlk}_{\epsilon\delta\,[\beta}V^{i}{}_{\gamma]\sigma}\dot{u}^{\sigma} - 2C_{hlk|r}B^{hlk}_{\epsilon\delta\,[\beta}V^{r}{}_{\gamma]\,\sigma}\dot{u}^{\sigma}$$
$$- 2C_{hlk|r}\dot{x}^{r}\,B^{lk}_{\delta\,[\beta}V^{h}{}_{\gamma]\epsilon}. \qquad \text{...(2.5)}$$

Considering the Berwald derivative of (2.4) with respect to u^{θ} and using (1.5) we have

$$H_{\epsilon\delta\beta\gamma\,(\theta)} = H_{hlkj\,(\theta)}\,B^{hlkj}_{\epsilon\delta\beta\gamma} + H_{hlkj}B^{lkj}_{\delta\beta\gamma}V^{h}{}_{\epsilon\theta} + H_{hlkj}B^{hkj}_{\epsilon\beta\gamma}\,V^{l}{}_{\delta\theta}$$
$$+ H_{hlkj}B^{hlj}_{\epsilon\delta\gamma}V^{k}{}_{\beta\theta} + H_{hlkj}B^{hlk}_{\epsilon\delta\beta}V^{j}{}_{\gamma\theta} + [(\overset{*}{M}{}^{i}\,\overset{*}{M}_{i})\,(g_{\beta\epsilon}g_{\delta\gamma}$$
$$-g_{\gamma\epsilon}g_{\delta\beta}) + 2M_{\delta}k_{n}^{2}\,(g_{\epsilon\beta}g_{\sigma\gamma}-g_{\epsilon\gamma}g_{\sigma\beta})\,\dot{u}^{\sigma}+P_{\epsilon\delta\beta\gamma}]_{\,(\theta)} \qquad \text{....(2.6)}$$

A direct calculation gives

$$\frac{\partial H_{hlkj}}{\partial u^{\theta}} = \frac{\partial H_{hlkj}}{\partial x^{m}}\,B_{\theta}{}^{m} + \frac{\partial H_{hlkj}}{\partial \dot{x}^{m}}\,B^{m}_{\sigma\theta}\dot{u}^{\sigma}\,. \qquad \text{...(2.7)}$$

And by definition we have

$$H_{hlkj\,(\theta)} = \frac{\partial H_{hlkj}}{\partial u^{\theta}} - \frac{\partial H_{hlkj}}{\partial \dot{u}^{\lambda}}\,\frac{\partial G^{\lambda}}{\partial \dot{u}^{\theta}} - (H_{rlkj}G^{r}{}_{hm}$$
$$+ H_{hrkj}G^{r}{}_{lm} + H_{hlrj}G^{r}{}_{km} + H_{hlkr}G^{r}{}_{jm})\,B_{\theta}{}^{m}. \qquad \text{...(2.8)}$$

After making necessary substitution from (2.7) in (2.8) and simplifying we get

$$H_{hlkj\,(\theta)} = \Big[\frac{\partial H_{hlkj}}{\partial x^{m}} - \frac{\partial H_{hlkj}}{\partial \dot{x}^{i}}\,\frac{\partial G^{i}}{\partial \dot{x}^{m}} - (H_{hrlk}G^{r}{}_{hm}$$
$$+ H_{hrkj}G^{r}{}_{lm} + H_{hlrj}G^{r}{}_{km} + H_{hlk\,r}G^{r}{}_{jm})\Big]\,B_{\theta}{}^{m}$$
$$+ \frac{\partial H_{hlkj}}{\partial \dot{x}^{i}}\left(\frac{\partial G^{i}}{\partial \dot{x}^{m}}\,B_{\theta}{}^{m} + B^{i}{}_{\sigma\theta}\dot{u}^{\sigma} - \frac{\partial G^{\lambda}}{\partial \dot{u}^{\theta}}\,B^{i}{}_{\lambda}\right), \qquad \text{...(2.9)}$$

which after using (1.9) can be expressed in the form:

$$H_{hlkj\,(\theta)} = H_{hlkj\,(m)}\,B_{\theta}{}^{m} + \frac{\partial H_{hlkj}}{\partial \dot{x}^{i}}\,V^{i}{}_{\sigma\theta}\dot{u}^{\sigma}.$$

After substituting this value of $H_{hlkj\,(\theta)}$ in (2.6) we get

$$H_{\epsilon\delta\beta\gamma\,(\theta)} = \left(H_{hlkj\,(m)}\,B_{\theta}{}^{m} + \frac{\partial H_{hlkj}}{\partial \dot{x}^{r}}\,V^{r}{}_{\sigma\theta}\dot{u}^{\sigma}\right)B^{hlkj}_{\epsilon\delta\beta\gamma}$$
$$+ [(\overset{*}{M}{}^{i}\,\overset{*}{M}_{i})\,(g_{\beta\epsilon}g_{\delta\gamma} - g_{\gamma\epsilon}g_{\delta\beta}) + 2M_{\delta}k_{n}^{2}\,(g_{\epsilon\beta}g_{\sigma\gamma} - $$

(equation continued on p. 501)

$$- g_{\epsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\epsilon\delta\beta\gamma}] (\theta) + [H_{hlkj} B^{lkj}_{\delta\beta\gamma} V^{h}_{\epsilon\theta}$$

$$+ H_{hlkj} B^{hkj}_{\epsilon\beta\gamma} V^{l}_{\delta\theta} + H_{hlkj} B^{hlj}_{\epsilon\delta\gamma} V^{k}_{\beta\theta} + H_{hlkj} B^{hlk}_{\epsilon\delta\beta} V^{j}_{\gamma\theta}]. \quad \ldots(2\cdot 10)$$

These are Gauss characteristic equations of a umbilical hypersurface immersed in a Finsler space. In the forming sections we shall use these equations to study the geometrical properties of the hypersurface.

3. Recurrent Finsler Spaces

If the curvature tensor H_{hlkj} of space F_n satisfies the relation

$$H_{hlkj(m)} = a_m H_{hlkj}, \quad \ldots(3\cdot 1)$$

where a_m is a non-zero vector, then F_n is called a recurrent Finsler space in the sense of Berwald. In the remaining part of this paper F_n is assumed to be recurrent.

A calculation based on eqns. (2.4), (2.10) and (3.1) gives

$$H_{\epsilon\delta\beta\gamma\,(\theta)} = a_m B_{\theta}{}^{m} [H_{\epsilon\delta\beta\gamma} - (\overset{*}{M}{}^{i} \overset{*}{M}_{i}) (g_{\epsilon\beta} g_{\delta\gamma} - g_{\epsilon\gamma} g_{\delta\beta}) - 2M_{\delta} k^{2}_{n} (g_{\epsilon\beta} g_{\sigma\gamma}$$

$$- g_{\epsilon\gamma} g_{\sigma\beta}) - P_{\epsilon\delta\beta\gamma}] + [(\overset{*}{M}{}^{i} \overset{*}{M}_{i}) (g_{\epsilon\beta} g_{\delta\gamma} - g_{\epsilon\gamma} g_{\delta\beta})$$

$$+ 2M_{\delta} k^{2}_{n} (g_{\epsilon\beta} g_{\sigma\gamma} - g_{\epsilon\gamma} g_{\sigma\beta}) + P_{\epsilon\delta\beta\gamma}]_{(\theta)}$$

$$+ \frac{\partial H_{hlkj}}{\partial \dot{x}^{r}} V^{r}{}_{\sigma\theta} \dot{u}^{\sigma} B^{hlkj}_{\epsilon\delta\beta\gamma} + H_{hlkj} (V^{h}{}_{\epsilon\theta} B^{lkj}_{\delta\beta\gamma}$$

$$+ V^{l}_{\delta\theta} B^{hkj}_{\epsilon\beta\gamma} + V^{k}{}_{\beta\theta} B^{hlj}_{\epsilon\delta\gamma} + V^{j}_{\gamma\theta} B^{hlk}_{\epsilon\delta\beta}). \quad \ldots(3\cdot 2)$$

We shall consider the following two cases:

(A) The recurrence vector field a_m is not normal to F_{n-1}. In other words

$$a_m B_{\theta}{}^{m} = a_{\theta} \neq 0.$$

(B) The recurrence vector field a_m is normal to F_{n-1},

i.e.,

$$a_m B_{\theta}{}^{m} = a_{\theta} = 0.$$

Case A

We assume that the hypersurface F_{n-1} is not totally geodesic. Defining

$$T_{\epsilon\delta\beta\gamma} = H_{\epsilon\delta\beta\gamma} - (\overset{*}{M}{}^{i} \overset{*}{M}_{i}) (g_{\epsilon\beta} g_{\delta\gamma} - g_{\epsilon\gamma} g_{\delta\beta}) - 2M_{\delta} k_{n}{}^{2} (g_{\epsilon\beta} g_{\sigma\gamma} - g_{\epsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma}$$

$$- P_{\epsilon\delta\beta\gamma} \quad \ldots(3\cdot 3)$$

and using condition (A) and eqn. (3·2) we obtain

$$T_{\epsilon\delta\beta\gamma\,(\theta)} = a_\theta T_{\epsilon\delta\beta\gamma} + \frac{\partial H_{hlkj}}{\partial \dot{x}^r} V^r_{\sigma\theta} \dot{u}^\sigma B^{hlkj}_{\epsilon\delta\beta\gamma}$$
$$+ H_{hlkj} (V^h_{\epsilon\theta} B^{lkj}_{\delta\beta\gamma} + V^l_{\delta\theta} B^{hkj}_{\epsilon\beta\gamma} + V^k_{\beta\theta} B^{hlj}_{\epsilon\delta\gamma} + V^j_{\gamma\theta} B^{hlk}_{\epsilon\delta\beta}). \quad \ldots(3\cdot4)$$

This equation and the fact $\Omega_{\sigma\theta} \neq 0$ lead to the following theorem:

Theorem 3·1—If a non-totally geodesic umbilical hypersurface F_{n-1} is immersed in a recurrent Finsler space whose recurrence vector field a_m is not normal to F_{n-1} then $T_{\epsilon\delta\beta\gamma}$ is recurrent with the recurrence vector $a_\theta = a_m B_\theta{}^m$ iff

$$\frac{\partial H_{hlkj}}{\partial \dot{x}^r} V^r_{\sigma\theta} \dot{u}^\sigma B^{hlkj}_{\epsilon\delta\beta\gamma} + H_{hlkj} (V^h_{\epsilon\theta} B^{lkj}_{\delta\beta\gamma}$$
$$+ V^l_{\delta\theta} B^{hkj}_{\epsilon\beta\gamma} + V^k_{\beta\theta} B^{hlj}_{\epsilon\delta\gamma} + V^j_{\gamma\theta} B^{hlk}_{\epsilon\delta\beta}) = 0. \quad \ldots(3\cdot5)$$

Defining

$$J_{\epsilon\delta\beta\gamma} = (\overset{*}{M}{}^i \overset{*}{M}_i) (g_{\epsilon\beta} g_{\delta\gamma} - g_{\epsilon\gamma} g_{\delta\beta}) + 2M_\delta k_n{}^2$$
$$\times (g_{\epsilon\beta} g_{\sigma\gamma} - g_{\epsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\epsilon\delta\beta\gamma} \quad \ldots(3\cdot6)$$

and using (3·3) we have

$$H_{\epsilon\delta\beta\gamma} = T_{\epsilon\delta\beta\gamma} + J_{\epsilon\delta\beta\gamma}. \quad \ldots(3\cdot7)$$

This equation reveals that $H_{\epsilon\delta\beta\gamma}$ is recurrent with the recurrence vector field a_θ provided that the terms $T_{\epsilon\delta\beta\gamma}$ and $J_{\epsilon\delta\beta\gamma}$ are recurrent with the same recurrence vector field a_θ. This fact and Theorem 3·1 proves the following:

Theorem 3·2—The sufficient condition that a non-totally geodesic umbilical hypersurface, immersed in a recurrent space F_n with the recurrence vector field a_m not normal to the hypersurface, be recurrent with the recurrence vector field a_θ are that the relation (3·5) holds and $J_{\epsilon\delta\beta\gamma}$ is recurrent with the recurrence vector field a_θ.

Case B

Here we assume that the recurrence vector field a_m is normal to F_{n-1}. The Bianchi identities for F_n are given by

$$H^i_{hkj\,(r)} + H^i_{hrk\,(j)} + H^i_{hjr\,(k)} + H^m_{kj} G^i_{mhj} + H^m_{rk} G^i_{mhj} + H^m_{jr} G^i_{mhk} = 0. \quad \ldots(3\cdot8)$$

It is also assumed that the space F_n is affinely connected then

$$C_{irk|h} = 0 \text{ and } G^i_{mhr} = 0 \quad \text{(Rund 1959, pp. 81, 82).} \quad \ldots(3\cdot9)$$

This condition reduces eqn. (3·8) to the form

$$H^i_{hkj(r)} + H^i_{hrk(j)} + H^i_{hjr(k)} = 0. \qquad ...(3\cdot10)$$

Multiplying the above by g_{il} and using the facts

$$g_{il(r)} = 0$$

we have

$$H_{hlkj(r)} + H_{hlrk(j)} + H_{hljr(k)} = 0. \qquad ...(3\cdot11)$$

The condition (3·1) for recurrent space F_n will reduce this equation to the form

$$a_r H_{hlkj} + a_j H_{hlrk} + a_k H_{hljr} = 0. \qquad ...(3\cdot12)$$

Multiplying the above equation by $B^{hlkj}_{\epsilon\delta\beta\gamma}$ and using the condition (B) and $a_r \neq 0$, we get

$$H_{hlkj} B^{hlkj}_{\epsilon\delta\beta\gamma} = 0. \qquad ...(3\cdot13)$$

Equations (3·13) and (2·4) give

$$H_{\epsilon\delta\beta\gamma} = (\overset{*}{M}{}^i \overset{*}{M}_i)(g_{\epsilon\beta} g_{\delta\gamma} - g_{\epsilon\gamma} g_{\delta\beta}) + 2M_\delta k_n^2 (g_{\epsilon\beta} g_{\sigma\gamma} - g_{\epsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\epsilon\delta\beta\gamma}. \qquad ...(3\cdot14)$$

This equation gives the following:

Theorem 3·3—If an umbilical hypersurface F_{n-1} is immersed in an affinely connected recurrent space F_n whose recurrence vector field a_m is normal to F_{n-1} then the tensor $T_{\epsilon\delta\beta\gamma}$ defined by (3·3) vanishes.

Further, eqns. (2·5) and (3·9) give

$$P_{\epsilon\delta\beta\gamma} = 2M_{lh} B^{hl}_{\delta\lambda} (\Lambda^\lambda{}_{\beta\sigma} \Omega_{\epsilon\gamma} - \Lambda^\lambda{}_{\gamma\sigma} \Omega_{\epsilon\beta}) \dot{u}^\sigma + 2\Lambda^\alpha{}_{\epsilon\,[\beta\,(\gamma)]} g_{\alpha\delta} - 2g_{\phi\delta} \Lambda^\sigma{}_{\epsilon\,[\beta} \times (\Lambda^\phi{}_{\gamma]\,\sigma} + C^\phi{}_{\gamma]\,\sigma|\rho} \dot{u}^\rho). \qquad ...(3\cdot15)$$

Therefore, eqns. (3·14) and (3·15) give the following:

Theorem 3·4—If an umbilical hypersurface F_{n-1} is immersed in an affinely connected recurrent space F_n whose recurrence vector field a_m is normal to F_{n-1} then the hypersurface F_{n-1} is Minkowskian (i.e., $H_{\epsilon\delta\beta\gamma} = 0$) iff

$$(\overset{*}{M}{}^i \overset{*}{M}_i)(g_{\epsilon\beta} g_{\delta\gamma} - g_{\epsilon\gamma} g_{\delta\beta}) + 2M_\delta k_n^2 (g_{\epsilon\beta} g_{\sigma\gamma} - g_{\epsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + 2M_{lh} B^{hl}_{\delta\lambda} (\Lambda^\lambda{}_{\beta\sigma} \Omega_{\epsilon\gamma} - \Lambda^\lambda{}_{\gamma\sigma} \Omega_{\epsilon\beta}) \dot{u}^\sigma + 2\Lambda^\alpha{}_{\epsilon\,[\beta(\gamma)]} g_{\alpha\delta} - 2g_{\delta\phi} \Lambda^\sigma{}_{\epsilon\,[\beta} (\Lambda^\phi{}_{\gamma]\,\sigma} + C^\phi{}_{\gamma]\,\sigma|\rho} \dot{u}^\rho) = 0. \qquad ...(3\cdot16)$$

If the hypersurface F_{n-1} is of minimal variety then mean curvature vector will vanish and eqns. (2·3) and (2·1) will give

$$k_n = 0, \quad \Omega_{\alpha\beta} = 0. \qquad ...(3\cdot17)$$

Equation (1.5) will now yield

$$\Lambda^{\epsilon}_{\beta\gamma} = 0. \qquad \ldots(3.18)$$

Theorem 3.4 and eqns. (3.17) and (3.18), therefore, give the following theorem:

Theorem 3.5—If a umbilical hypersurface F_{n-1} is of minimal variety and is immersed in an affinely connected recurrent space F_n whose recurrence vector field is normal to F_{n-1} then the space F_{n-1} is Minkowskian.

Multiplying the relation (3.14) by $\dot{u}^{\delta}$ and using (3.15) and the fact $M_{i|h}\dot{x}^{h} = 0$, we get

$$H_{\epsilon\delta\beta\gamma}\dot{u}^{\delta} = (\overset{*}{M}{}^{i}\overset{*}{M}_{i})(g_{\epsilon\beta}\, g_{\delta\gamma} - g_{\epsilon\gamma}\, g_{\delta\beta})\,\dot{u}^{\delta}$$
$$+ 2\Lambda^{\alpha}_{\epsilon\,[\beta(\gamma)]}\, g_{\alpha\delta}\dot{u}^{\delta} - 2g_{\phi\delta}\dot{u}^{\delta}\, \Lambda^{\sigma}_{\epsilon\,[\beta}\, (\Lambda^{\phi}{}_{\gamma]\,\sigma} + C^{\phi}{}_{\gamma]\,\sigma|\rho}\dot{u}^{\rho}). \qquad \ldots(3.19)$$

Since the hypersurface F_{n-1} of constant curvature is characterized by the equation

$$H_{\epsilon\delta\beta\gamma}\dot{u}^{\delta} = K(g_{\epsilon\beta}\, g_{\delta\gamma} - g_{\epsilon\gamma}\, g_{\delta\beta})\,\dot{u}^{\delta}$$

we have, therefore, established the following theorem:

Theorem 3.6—If a umbilical hypersurface F_{n-1} is immersed in an affinely connected recurrent Finsler space F_n whose recurrence vector field is normal to F_{n-1}, then the necessary and sufficient condition that F_{n-1} be a space of constant curvature is that

$$\Lambda^{\alpha}{}_{\epsilon[\beta(\gamma)]} - \Lambda^{\sigma}{}_{\epsilon[\beta}\,(\Lambda^{\alpha}{}_{\gamma]\sigma} + C^{\alpha}{}_{\gamma]\,\sigma|\rho}\dot{u}^{\rho}) = 0.$$

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