ON SOME TYPE OF AFFINE MOTION IN BI-RECURRENT FINSLER SPACES—II

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In this paper the author has continued his study on affine motion in a recurrent Finsler Space of second order. Three different cases have been discussed in detail and it has been shown that the affine motion of the above form does not admit a case characterized by $K_{hj} v^h = 0$.

1. INTRODUCTION

Let us consider an n-dimensional affinely connected and non-flat space F_n (Rund 1959) having a symmetric connection $\Gamma^i_{jk}(x, \dot{\xi})$. The covariant derivative of any tensor field $T^i_j(x, \dot{\xi})$ with respect to Γ^i_{jk} is given by

$$T^i_{j;k} = \partial_k T^i_j + T^s_j \Gamma^i_{sk} - T^i_s \Gamma^s_{jk} + \dot{\partial}_h T^i_j \partial_k \dot{\xi}^h \qquad \ldots(1\cdot1)$$

The commutation formula involving the above covariant derivative is given by

$$2T^i_{j[;h;k]} = T^s_j \tilde{K}^i_{shk} - T^i_s \tilde{K}^s_{jhk}, \qquad \ldots(1\cdot2)$$

where

$$\tilde{K}^i_{hjk}(x, \dot{\xi}) \overset{\text{def}}{=\!=} 2\{\partial_{[k} \Gamma^i_{j]h} + \dot{\partial}_s \Gamma^i_{h[j} \partial_{k]} \dot{\xi}^s + \Gamma^i_{m[k} \Gamma^m_{j]h}\} \qquad \ldots(1\cdot3)$$

is called relative curvature tensor field and satisfies the following relations (Rund 1959):

$$\tilde{K}^i_{hjk} = -\tilde{K}^i_{hkj} \qquad \ldots(1\cdot4)$$

$$(a)\ \tilde{K}^i_{hji} = \tilde{K}_{hj} \text{ and } (b)\ \tilde{K}^i_{ihj} = \tilde{K}_{hj} - \tilde{K}_{jh} \qquad \ldots(1\cdot5)$$

$$\tilde{K}^i_{hjk} + \tilde{K}^i_{jkh} + \tilde{K}^i_{khj} = 0 \qquad \ldots(1\cdot6)$$

and

$$\tilde{K}^i_{hjk;s} + \tilde{K}^i_{hks;j} + \tilde{K}^i_{hsj;k} = 0. \qquad \ldots(1\cdot7)$$

$2A_{[hk]} = A_{hk} - A_{kh}$

$\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$ and $\partial_i \equiv \partial/\partial x^i$.

Now, let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\ dt, \qquad \ldots(1\cdot 8)$$

where $v^i(x)$ is any covariant vector independent of directional arguments and dt is an infinitesimal point constant. By virtue of the above point transformation and covariant derivative, we have (Yano 1957)

$$\mathcal{L}u\, T^i{}_j = T^i{}_{j;h}\, v^h + T^i{}_k\, v^k{}_{;j} - T^h{}_j\, v^i{}_{;h} \qquad \ldots(1\cdot 9)$$

and

$$\mathcal{L}v\, \Gamma^i{}_{jk}(x, \xi) = v^i{}_{;j;k} + \tilde{K}^i{}_{jkh}\, v^h. \qquad \ldots(1\cdot 10)$$

If the relative curvature tensor field $\tilde{K}^i{}_{hjk}(x, \xi)$ satisfies the relation:

$$\tilde{K}^i{}_{hjk;s;m} = t_{sm}\, \tilde{K}^i{}_{hjk}, \qquad \ldots(1\cdot 11)$$

where t_{sm} means in general a non-zero and non-symmetric covariant tensor, then the space is called bi-recurrent Finsler space.

In a previous paper the present author (Kumar 1976) has studied an affine motion of special types and investigated the essential properties of the space. As a continuation of our study, in this paper, we shall try to investigate on the space admitting an affine motion (1·8) of bi-recurrent form characterized by

$$v^i{}_{;m;s} = d_{ms}\, v^i, \qquad \ldots(1\cdot 12)$$

where, in general, we assume that $t_{ms} \neq d_{ms}$. Being (1.8) an affine motion, it is characterized by

$$\mathcal{L}v\, \Gamma^i{}_{jk} = v^i{}_{;j;k} + \tilde{K}^i{}_{jkh}\, v^h = 0 \qquad \ldots(1\cdot 13)$$

from which as its integrability condition, we can have

$$\mathcal{L}v\, \tilde{K}^i{}_{hjk} = 0. \qquad \ldots(1\cdot 14)$$

According to the method adopted by Takano and Imai (1972) let us introduce a non-symmetric tensor p^{sm} such that

$$v^i{}_{;h} = \tilde{K}^i{}_{hsm}\, p^{sm}. \qquad \ldots(1\cdot 15)$$

Thus, in view of the above equation, the condition (1·14) can be written like (Kumar 1976 *a*)

$$\tilde{K}^i{}_{hjk;s}\, v^s = B\tilde{K}^i{}_{hjk}, \qquad \ldots(1\cdot 16)$$

where

$$B \overset{\text{def}}{=\!=} P_{sm}\, p^{sm} \qquad \ldots(1\cdot 17)$$

and

$$P_{sm} \overset{\text{def}}{=\!=} (t_{sm} - t_{ms}). \qquad \ldots(1\cdot 18)$$

2. Occurrence of Two Cases

Under the existence of affine motion (1·8) of the form (1·12), equation (1·13) reduces to,

$$d_{jk}\, v^i + \tilde{K}^i{}_{jkh}\, v^h = 0. \qquad \ldots(2\cdot1)$$

Transvecting the last formula by v^k and taking care of the facts that $\tilde{K}^i{}_{hsm}\, v^s\, v^m = 0$ and $v^i \neq 0$, we get

$$d_{jk}\, v^k = 0. \qquad \ldots(2\cdot2)$$

On the other hand, contracting the formula (2·1) with respect to the indices i and k and using eqns. (1·4), (1·5*a*) and (2·2), we have

$$\tilde{K}_{jh}\, v^h = 0. \qquad \ldots(2\cdot3)$$

In view of eqn. (1·5*b*), contracting the formula (1·13), with respect to the indices i and j, we obtain

$$\lambda_{;k} = (\tilde{K}_{kh} - \tilde{K}_{hk})\, v^h, \qquad \ldots(2\cdot4)$$

where

$$\lambda \overset{\text{def.}}{=\!=} v^i{}_{;i}. \qquad \ldots(2\cdot5)$$

Now, with the help of eqns. (2·3) and (2·4), we can get

$$\lambda_{;k} = -\, \tilde{K}_{hk}\, v^h. \qquad \ldots(2\cdot6)$$

Differentiating (2·6) covariantly twice with respect to x^m x^s and noting the eqns. (1·11) and (1·12), we have

$$\lambda_{;k;m;s} = -\, t_{ms}\, \tilde{K}_{hk}\, v^h - \tilde{K}_{hk;m}\, v^h{}_{;s} - \tilde{K}_{hk;s}\, v^h{}_{;m} - \tilde{K}_{hk}\, d_{ms}\, v^h. \qquad \ldots(2\cdot7)$$

Now, commutating the last formula with respect to the indices m and s and using the commutation formula (1·2) and equation (1·18), we obtain

$$(P_{sm} + d_{sm} - d_{ms})\, \tilde{K}_{hk}\, v^h = \lambda_{;h}\, \tilde{K}^h{}_{kms}. \qquad \ldots(2\cdot8)$$

Next, let us transvect the formula (2·6) by v^k and using eqn. (2·3), we get

$$\lambda_{;k}\, v^k = 0. \qquad \ldots(2\cdot9)$$

Again, transvecting the formula (2·8) by v^s and taking care of eqns. (2·1) and (2·9), we can have

$$(P_{sm} + d_{sm} - d_{ms})\, v^s\, \tilde{K}_{hk}\, v^h = 0. \qquad \ldots(2\cdot10)$$

In this way, we arrive at the following conclusion:

Conclusion—If a bi-recurrent space under consideration admits and affine motion (1·8) of bi-recurrent form (1·12), then the following two cases must be considered:

$$\text{(i) } (P_{sm} + d_{sm} - d_{ms})\, v^s = 0, \quad \text{(ii) } \tilde{K}_{hk}\, v^h = 0. \qquad \ldots(2\cdot11)$$

In what follows, we shall investigate on these cases one by one.

3. The Case (i)

In view of eqn. (2·2), the former case [2·11 (i)] may be replaced by

$$\alpha_m + d_{sm}\, v^s = 0, \qquad \ldots(3\cdot1)$$

where

$$\alpha_m \overset{\text{def}}{=\!=} B_{sm}\, v^s. \qquad \ldots(3\cdot2)$$

Owing to the following relation:

$$d_{sm}\, v^s = v^s{}_{;s;m} = \lambda_{sm} \qquad \ldots(3\cdot3)$$

the formula (3·1) may be replaced by

$$\alpha_m = -\lambda_{;m}. \qquad \ldots(3\cdot4)$$

With the help of eqns. (2·6) and (3·4), we can conclude

$$\alpha_m = K_{hm}\, v^h. \qquad \ldots(3\cdot5)$$

Now, introducing the last formula into eqn. (3·1), we have

$$(\tilde{K}_{sm} + d_{sm})\, v^s = 0. \qquad \ldots(3\cdot6)$$

Next, let us differentiate (3·6) covariantly twice with respect to x^k and using eqns. (1·11), (1·12) and (3·6), we get

$$(d_{sm;h;k} + t_{hk}\tilde{K}_{sm})\, v^s + (\tilde{K}_{sm;k} + d_{sm;k})\; v^s{}_{;h} + (\tilde{K}_{sm;h} + d_{sm;h})\; v^s{}_{;k} = 0. \qquad \ldots(3\cdot7)$$

From this formula, considering a commutation on h and k, we can find

$$-\, d_{nm}\,(v^n{}_{;h;k} - v^n{}_{;k;h}) - d_{sn}\tilde{K}^n{}_{mhk}v^s + P_{hk}\tilde{K}_{sm}v^s = 0 \qquad \ldots(3\cdot8)$$

i.e.,

$$-\, d_{kh}d_{nm}v^n + d_{kh}d_{nm}v^n - d_{sm}\tilde{K}^n{}_{mhk}v^s + P_{hk}\tilde{K}_{sm}v^s = 0 \qquad \ldots(3\cdot9)$$

where we have used the commutation formula (1·2) and eqns. (1·12) and (1·18).

On the other hand, by virtue of eqns. (3.1), (3.2) and (3.7), the relation (3.9) reduces to

$$(P_{hk} + d_{hk} - d_{kh})\,\alpha_m + \tilde{K}^n{}_{mhk}\,\alpha_n = 0. \qquad ...(3.10)$$

Now, transvecting the formula (3.5) by v^m and taking care of the fact $\tilde{K}_{kj} v^k v^j = 0$ and eqn. (3.4), we obtain

$$\alpha_m v^m = \lambda_{;m} v^m = 0. \qquad ...(3.11)$$

On the other hand, with the help of eqns. (1.4), (1.12) and (1.13), we can construct

$$\tilde{K}^i{}_{hkj} v^k = -\tilde{K}^i{}_{hjk} v^k = v^i{}_{;h;j} = d_{hj} v^i. \qquad ...(3.12)$$

Thus, transvecting the formula (3.10) by v^k and using eqns. (3.1), (3.11) and (3.12), we have

$$(P_{hk} + d_{hk} - d_{kh})\, v^k \alpha_m = 0. \qquad ...(3.13)$$

Hence, when and only when $\alpha_m \neq 0$, we have our preliminary condition [2.11 (i)]. That is under $\alpha_m \neq 0$, the preliminary condition and eqn. (3.6) are equivalent to each other. Here, we have to take care of the fact that $\alpha_m \neq 0$ means $\lambda_{;m} \neq 0$. In what follows, we shall assume this fact. The case of $\lambda_{;m} = 0$ will be discussed in the next section.

In view of eqns. (1.5*a*) and (1.15), we can get

$$\mathcal{L}v \tilde{K}_{hk} = 0. \qquad ...(3.14)$$

By virtue of eqns. (1.9) and (1.16), the last formula can be re-written like

$$B\tilde{K}_{hj} + \tilde{K}_{sj} v^s{}_{;h} + \tilde{K}_{hs} v^s{}_{;j} = 0. \qquad ...(3.15)$$

Now, with the help of eqns. (1.12) and (2.3), we can conclude

$$\tilde{K}_{hs} v^s{}_{;j;m} = \tilde{K}_{hs} d_{jm} v^s = d_{jm} \tilde{K}_{hs} v^s = 0. \qquad ...(3.16)$$

Next, taking the covariant derivative of (3.15) with respect to x^m and taking notice of eqns. (1.12) and (3.16), we find

$$B_{;m}\tilde{K}_{hj} + B\tilde{K}_{hj;m} + \tilde{K}_{sj;m} v^s{}_{;h} + \tilde{K}_{sj}\, d_{hm} v^s + \tilde{K}_{hs;m} v^s{}_{;j} = 0. \qquad ...(3.17)$$

In view of eqns. (3.4) and (3.5), the last result may be replaced by

$$B_{;m}\tilde{K}_{hj} + B\tilde{K}_{hj;m} + \tilde{K}_{sj;m} v^s{}_{;h} + \tilde{K}_{hs;m} v^s{}_{;j} - \lambda_{;j} = 0. \qquad ...(3.18)$$

Now, differentiating last relation covariantly with respect to x^s and taking care of eqns. (1.11), (1.12) and (3.15), we obtain

$$B_{;m;s}\tilde{K}_{hj} + B_{;s}\tilde{K}_{hj;m} + B_{;m}\tilde{K}_{hj;s} + \tilde{K}_{rj;m} d_{hs} v^r + \tilde{K}_{hr;m} d_{js} v^r - d_{hm;s}\lambda_{;j}$$
$$- d_{hm}\lambda_{;j;s} = 0. \qquad ...(3.19)$$

With the help of eqns. (1·5*a*) and (1·16), we can construct

$$\tilde{K}_{hj;s}v^s = B\tilde{K}_{hj}. \qquad ...(3\cdot20)$$

Now, let us transvect the formula (3·19) by v^m and taking notice of the last equation we have

$$(B_{;m;s}v^m + BB_{;s})\,\tilde{K}_{hj} + d_{hs}\,(-\lambda_{;j}) - v^m d_{hm;s}\lambda_{;j}) = 0, \qquad ...(3\cdot21)$$

where, we have used the fact

$$\mathcal{L}_v B = B_{;m}v^m = 0. \qquad ...(3\cdot22)$$

(see Kumar 1976*a*, § 3)

The equation (3·21) can also be re-written like

$$(B_{;m;s}v^m + BB_{;s})\,\tilde{K}_{hj} = (Bd_{hs} + d_{hm;s}v^m)\,\lambda_{;j}. \qquad ...(3\cdot23)$$

Under the basic assumption $\lambda_{;j} \neq 0$, the above formula suggests the resolvability of Ricci tensor $\tilde{K}_{hj}(x, \xi)$ of the form

$$\tilde{K}_{hj} = \eta_h\lambda_{;j}, \qquad ...(3\cdot24)$$

where $\eta_h(x)$ is any covariant vector.

In the following lines we shall discuss on this point. For this purpose let us suppose that

$$B_{;m;s}v^m + BB_{;s} = 0. \qquad ...(3\cdot25)$$

holds and hence the desired decomposition (3·24) of $\tilde{K}_{hj}(x, \xi)$ is impossible. As, we have

$$\mathcal{L}vB = B_{;m}v^m = 0. \qquad ...(3\cdot26)$$

Differentiating the equation (3·26) covariantly with respect to x^s, we get

$$B_{;m;s}v^m + B_{;m}v^m{}_{;s} = 0. \qquad ...(3\cdot27)$$

Thus, in view of the above formula, the supposition (3·25) reduces to

$$BB_{;s} = B_{;m}v^m{}_{;s} \qquad ...(3\cdot28)$$

Now, introducing the last formula into the left-hand side of equation (3·23), we obtain

$$(Bd_{sh} + d_{hm;s}v^m)\,\lambda_{;j} = 0. \qquad ...(3\cdot29)$$

For the assumption that $\lambda_{;j} \neq 0$, the last formula may be replaced by

$$Bd_{hs} + d_{hm;s}v^m = 0. \qquad ...(3\cdot30)$$

Next, by virtue of equation (2·2), the above formula takes the form

$$Bd_h = d_{hm}v^m{}_{;s}. \qquad ...(3\cdot31)$$

On the other hand, transvecting the last formula v^h and noting equation (3·3) we get

$$B\lambda_{;s} = \lambda_{;m}v^m{}_{;s}. \qquad \text{...(3·32)}$$

Next, let us multiply the last result by a non-vanishing scalar function B and, differentiating the resulting equation with respect to x^h, we find

$$2BB_{;h}\lambda_{;s} + B^2\lambda_{;s;h} = B\lambda_{;m;h}v^m{}_{;s} + B_{;h}\lambda_{;m}v^m{}_{;s}, \qquad \text{...(3·33)}$$

where we have used eqns. (1·12) and (3·11). With the help of eqns. (1·2) and (1·17), we can get

$$v^i{}_{;h}v^h = v^h\tilde{K}^i{}_{hsm}p^{sm} = p^{sm}(v^i{}_{;s;m} - v^i{}_{;m;s}) = \tilde{\Omega}v^i, \qquad \text{...(3·34)}$$

where

$$\tilde{\Omega} \underset{=\!=}{\text{def}} p^{sm}(d_{sm} - d_{ms}). \qquad \text{...(3·35)}$$

Transvecting the formula (3·33) by v^s and taking notice of eqns. (3·11) and (3·34), we find

$$B^2\lambda_{;s;h}v^s = B\tilde{\Omega}v^m\lambda_{;m;h}. \qquad \text{...(3·36)}$$

On the other hand with the help of equation (3·11), we can construct

$$\lambda_{;s;h}v^s = (\lambda_{;s}v^s)_{;h} - \lambda_{;s}v^s{}_{;h} = -\lambda_{;s}v^s{}_{;h}. \qquad \text{...(3·37)}$$

Hence, introducing the last formula into equation (3·36), we have

$$B(B - \tilde{\Omega})\lambda_{;s}v^s{}_{;h} = 0. \qquad \text{...(3·38)}$$

However, in general, $B \neq 0$ and $B \neq \tilde{\Omega}$, so from the above equation we can find

$$\lambda_{;s}v^s{}_{;h} = 0. \qquad \text{...(3·39)}$$

Now, substituting the last result into the right-hand side of eqn. (3·32) we get

$$B\lambda_{;s} = 0. \qquad \text{...(3·40)}$$

This contradicts with the basic assumption $\lambda_{;s} \neq 0$. By this reason we cannot institute the condition (3·28), say, we must have

$$B_{;m;s}v^m + B_{;m}v^m{}_{;s} \neq 0. \qquad \text{...(3·41)}$$

In this way, we have the following theorems:

Theorem 3·1—When the bi-recurrent Finsler space under consideration admits an affine motion (1·8) of bi-recurrent form (1·12), there exists case of $(P_{hk} + d_{hk} - d_{kh})\, v^k = 0$. In this case, under $\lambda_{;s} \neq 0$, Ricci tensor $\tilde{K}_{hj}\,(x, \xi)$ is resolved as $\eta_h\lambda_{;j}$ with use of the covariant vector η_h.

4. The Case (ii)

In this case, we have not only

$$\lambda_{;h} = v^s{}_{;s;h} = -\tilde{K}^s{}_{shm} v^m = -(-\tilde{K}_{hm} + \tilde{K}_{mh})\, v^m = 0, \quad \ldots(4.1)$$

followed from (1·13) and (2·3), but also

$$\lambda_{;h} = v^s{}_{;s;h} = d_{sh} v^h. \quad \ldots(4.2)$$

In this way, we find

$$d_{hj} v^j = d_{hj} v^h = 0. \quad \ldots(4.3)$$

On one hand, in the present space, the author (Kumar 1976*b*) has investigated the following formula:

$$P_{ms} \tilde{K}^s{}_{hjk} = t_{ks} \tilde{K}^s{}_{mjh} - t_{js} \tilde{K}^s{}_{mkh} - t_{kh} \tilde{K}_{mj} + t_{jh} \tilde{K}_{mk} - P_{jk} P_{ms}. \quad \ldots(4.4)$$

Transvecting the above result by v^m and noting eqns. (1·2), (2·3) and (3·2), we obtain

$$\alpha_s \tilde{K}^s{}_{hjk} = t_{ks} (v^s{}_{;j;h} - v^s{}_{;h;j}) - t_{js} (v^s{}_{;k;h} - v^s{}_{;h;k}) - P_{jk} \alpha_h. \quad \ldots(4.5)$$

In view of equation (1.12), the last formula reduces to

$$\alpha_s \tilde{K}^s{}_{hjk} = t_{ks} v^s (d_{js} - d_{hj}) - t_{js} v^s (d_{kh} - d_{hk}) - P_{jk} \alpha_h. \quad \ldots(4.6)$$

On the other hand, with help of eqns. (1·12), (1·13) and (3·11), we can conclude

$$\alpha_s \tilde{K}^s{}_{hjk} v^k = -\alpha_s v^s{}_{;h;j} = -\alpha_s v^s d_{hj} = 0. \quad \ldots(4.7)$$

Thus, transvecting the formula (4·6) by v^k and taking notice of eqns. (3·2), (4·2) and (4·6), we have

$$\alpha_h \alpha_j = (d_{hj} - d_{jh})\, t_{ks} v^k v^s = 0. \quad \ldots(4.8)$$

In the above equation, taking care of the symmetric property of the left-hand side on the indices h and j and the alternative property of the right-hand side on the same indices, that

$$t_{ks} v^k v^s (d_{hj} - d_{jh}) = 0 \text{ and } d_j = 0. \quad \ldots(4.9)$$

Consequently, there exist two cases to be considered. They are

$$\text{(i) } d_{hj} = d_{jh} \text{ and (ii) } \alpha_j = 0 \quad \ldots(4.10)$$

and

$$\text{(i) } t_{ks} v^k v^s = 0 \text{ and (ii) } \alpha_j = 0 \quad \ldots(4.11)$$

respectively.

By virtue of the first case [4.10(i)], we can conclude

$$v^i_{;h;j} - v^i_{;j;h} = v^i (d_{hj} - d_{jh}) = 0. \qquad ...(4.12)$$

i.e.,

$$\tilde{K}^i_{mhj} v^m = 0. \qquad ...(4.13)$$

Consequently, the vector field $v^i(x)$ may be considered to be degenerated into the contra-field. By this reason, the first case should be ruled out of our study.

Let us now consider the second case [4.11 (i)]. In view of eqns. (1.12) and (1.17), we can construct

$$v^m t_{ms} \tilde{K}^i_{hjk} = v^m \tilde{K}^i_{hjk;m;s} = (v^m \tilde{K}^i_{hjk;m})_{;s} - \tilde{K}^i_{hjk;m} v^m_{;s}$$

$$= (B\tilde{K}^i_{hjk})_{;s} - \tilde{K}^i_{hjk;m} v^m_{;s}. \qquad ...(4.14)$$

i.e.,

$$v^m t_{ms} \tilde{K}^i_{hjk} = B_{;s} \tilde{K}^i_{hjk} + B\tilde{K}^i_{hjk;s} - \tilde{K}^i_{hjk;m} v^m_{;s}. \qquad ...(4.15)$$

Transvecting the last formula by v^s and noting eqns. [4.11 (i)], (1.16), (3.34) and the fact that $B_{;s} v^s = B$, we get

$$B(B - \tilde{\Omega}) \tilde{K}^i_{hjk} = 0. \qquad ...(4.16)$$

As we have $B \neq 0$ and $B \neq \tilde{\Omega}$, the last formula may be replaced by

$$\tilde{K}^i_{hjk} = 0. \qquad ...(4.17)$$

But this is a contradiction because the space under consideration is a non-flat one. Therefore, the second case should be also excepted from our study. Thus, we can state the following:

Theorem 4.1—A bi-recurrent Finsler space having an affine motion (1.8) of bi-recurrent form (1.12) does not admit a case $\tilde{K}_{hj} v^h = 0$.

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