INTER-RELATION BETWEEN SOLVABLE SCALAR POTENTIALS FOR THE NON-RELATIVISTIC AND RELATIVISTIC WAVE EQUATIONS

by KAMAL S. RAJPUT*, *Department of Physics, Government Post-graduate College, Tikamgarh* 472001

(*Received* 29 *September* 1976)

Following the general methods developed by Bhattacharjie and Sudarshan (1962) and Bose (1963) a large number of solvable scalar potentials for the non-relativistic Schrödinger equation have been constructed from the transformation of linear second order differential equations. The same methods can be used to construct solvable scalar potentials for the Klein–Gordon and Dirac wave equations, if under suitable approximations, these relativistic wave equations are reduced to the radial forms similar to the one already used for the Schrödinger equation. In this paper an attempt is made to this effect. The author obtains a Schrödinger like radial wave equation to which both the Klein–Gordon and Dirac wave equations reduce under certain approximations. A particular second order differential equation can now be transformed to yield solvable scalar potentials for the non-relativistic and relativistic wave equations. A qualitative probe into the inter-relation between these potentials reveals some basic similarities and differences inherent in the wave equations.

1. INTRODUCTION

Extensive efforts have been made to construct solvable scalar potentials for the non-relativistic Schrödinger equation (Bargmann 1949, Morse and Feshbach 1953, Bhattacharjie and Sudarshan 1962, Bose 1963, Aly and Spector 1965, Lemieux and Bose 1969, Sharma and Varma 1970, Sharma 1970*a* and 1970*b*, Sharma and Singh 1971, Khan 1973, Rajput 1976). Similar efforts have been made to study the solutions of the relativistic Klein-Gordon and Dirac wave equations with scalar and vector potentials (Sengupta 1947, 1964, Stanciu 1966, Vasudevan *et al.*, 1967, Evans 1970, Soff *et al.*, 1973). Sengupta (1964) carried out an interesting study on the inter-relation between the solutions of the classical Hamilton-Jacobi equation, the Klein-Gordon wave equation and the Dirac wave equation with a class of electromagnetic fields. Vasudevan *et al.* (1967) have tried to obtain few solvable scalar potentials for the Klein-Gordon equation and the Dirac equation,

* Present address: Department of Physics, Maharaja College, Chhatarpur 471 001.

in the non-relativistic limits, and also probed their inter-relation with the potentials for the Schrödinger equation. As in the classical limit the relativistic wave equations reduce to their non-relativistic counterparts; it is of interest to probe deeper into the inter-relationship of solvable scalar potentials for the two classes of wave equations.

The present paper is based upon the proposition that the solvable scalar potentials for relativistic Klein-Gordon and Dirac wave equations may be constructed by following the general methods of Bhattacharjie and Sudarshan (1962) and Bose (1963); if the two relativistic wave equations, under suitable approximations, be transformed to some radial forms similar to the one already used for the non-relativistic Schrödinger equation.

In section 2 we give a brief resumé of the general methods developed by Bhattacharjie and Sudarshan (1962) and Bose (1963); used extensively to construct solvable potentials for the Schrödinger equation. A slight modification is suggested to suit the construction of scalar potentials for the relativistic wave equations in natural units ($\hbar = c = 1$).

In section 3, the Klein-Gordon equation is reduced to a radial form similar to the one already used for the Schrödinger equation. The transformation of the Dirac equation to the radial form of the Schrödinger equation poses certain problems concerning the conservation of parity. However, in weak interactions, where the parity is not necessarily conserved, Feynman (1958) has suggested a two-component formalism for the Dirac equation. Biedenharn (1972) has also suggested a two-component formalism on a similar ground. Following the suggestion of Feynman (1958) that the decoupling of the two coupled Dirac equations is only an algebraic problem and the proposal of Sengupta (1964) on the conditions under which the two coupled Dirac equations can be treated as simultaneous equations; we have reduced the coupled Dirac wave equations to a form similar to the radial Schrödinger equation. It is found that under the approximations, the Dirac equation reduces to the radial form of the Klein-Gordon equation already obtained. Thus, the limiting form of the two relativistic wave equations being one, we suggest the application of the Bhattacharjie and Sudarshan (1962) and Bose's (1963) methods to construct the solvable scalar potentials for them.

A particular second order differential equation can now be transformed to yield solvable scalar potentials for the non-relativistic as well as the relativistic wave equations. In section 4, a qualitative probe into the inter-relation between the potentials for the two classes of wave equations, reveals some basic similarities and differences inherent in the wave equations under study.

In section 5 we discuss the main conclusions with reference to their support in the literature. The energy dependence of the solvable potentials for the common radial form of the relativistic wave equations is discussed and its interpretation in terms of velocity dependent potentials is probed.

2. A Brief Resumé of the Methods to Construct Solvable Scalar Potentials

(I) *The Bhattacharjie and Sudarshan Method*

Bhattacharjie and Sudarshan (1962) considered a general second order differential equation in one variable in the form

$$\frac{d^2}{dz^2} u(z) + p(z) \frac{d}{dz} u(z) + q(z) u(z) = 0. \qquad \ldots(2\cdot 1)$$

On making the transformations

$$z = f(r),\ u(z) = g(r)\ \phi(r), g(r) \neq 0, \qquad \ldots(2\cdot 2)$$

the eqn. (2·1) is put in the form

$$\frac{d^2}{dr^2} \phi(r) + A(r) \frac{d}{dr} \phi(r) + B(r) \phi(r) = 0, \qquad \ldots(2\cdot 3)$$

where

$$A(r) = \frac{2}{g(r)} \frac{d}{dr} g(r) + P(r) \frac{d}{dr} f(r) - \frac{(d^2 f(r)/dr^2)}{(d/dr) f(r)}, \qquad \ldots(2\cdot 4a)$$

$$B(r) = \frac{d^2}{dr^2} g(r)/g(r) + Q(r) \left\{\frac{d}{dr} f(r)\right\}^2$$

$$+ \left\{\frac{dg(r)/dr}{g(r)}\right\} \left\{P(r) \frac{df(r)}{dr} - \frac{d^2 f(r)/dr^2}{df(r)/dr}\right\}, \qquad \ldots(2\cdot 4b)$$

where

$$P(r) = p\{f(r)\} \qquad \ldots(2\cdot 4c)$$

$$Q(r) = q\{f(r)\}. \qquad \ldots(2\cdot 4d)$$

For (2·3) to be of the form of the radial s-wave Schrödinger equation with the Scalar potential $V_s(r)$,

$$\frac{d^2}{dr^2} \phi(r) + [k^2 - V_s(r)]\ \phi(r) = 0, \qquad \ldots(2\cdot 5)$$

the necessary and sufficient conditions are

$$A(r) = 0,\ B(r) = k^2 - V_s(r), \frac{\partial}{\partial k} V_s(r) = 0. \qquad \ldots(2\cdot 6)$$

Thus, under suitable transformations a linear second order differential equation can be transformed to the Schrödinger equation (2·5) with the scalar potential $V_s(r)$.

(II) *The Bose Method*

Bose (1963) considers the normal form of the second order differential equation

$$v'' + I(z) v = 0 \qquad \ldots(2\cdot 7)$$

and substituting

$$z = z(x) \text{ and } v(z) = z'^{\frac{1}{2}} f(x), \qquad \text{...(2.8)}$$

obtains

$$f'' + [z'^2 I(z) + \tfrac{1}{2}\{z, x\}] f = 0; \qquad \text{...(2.9)}$$

where the Schwartzian derivative

$$\{z, x\} = \frac{d^2 \log z'}{dx^2} - \frac{1}{2}\left(\frac{d \log z'}{dx}\right)^2 . \qquad \text{...(2.10)}$$

If we consider

$$I_s(x) = z'^2 I(z) + \tfrac{1}{2}\{z, x\}, \qquad \text{...(2.11)}$$

for a one-dimensional Schrödinger equation (2.5) (for $r = \alpha x$)

$$I_s(x) = k^2 - V_s(x). \qquad \text{...(2.12)}$$

In both the methods I and II, the units used are $\hbar = 2m = 1$ such that the non-relativistic energy $E' = k^2$.

It has been verified repeatedly (Sharma, 1970*a*, 1970*b*; Khan 1973) that the two methods lead to the same set of potentials. For a one-dimensional Schrödinger equation (2.5) with $r = \alpha x$, the equality of V_s in (2.6) and (2.12) gives

$$B(x) = I_s(x). \qquad \text{...(2.13)}$$

In the treatment of relativistic wave equations, we use the natural units ($\hbar = c = 1$) with the relativistic energy $E = E' + m$. The Schrödinger equation (2.5) can be re-written as

$$\frac{d^2}{dr^2}\phi(r) + 2m[(E-m) - V_s(r)]\phi(r) = 0 \qquad \text{...(2.5a)}$$

and the necessary and sufficient conditions are written as

$$A(r) = 0,\ B(r) = 2m[(E-m) - V_s(r)],\ \frac{\partial V_s}{\partial k} = 0. \qquad \text{...(2.6a)}$$

Also, we have

$$I_s(x) = 2m\ [(E-m) - V_s(x)]. \qquad \text{...(2.12)}$$

3. Reduction of the Relativistic Wave Equations

(I) *The Klein-Gordon Wave Equation*

We consider the Klein-Gordon wave equation [Messiah 1970, (XX, 30), 886] in the form

$$\left[\left(i\frac{\partial}{\partial t} - e\Phi\right)^2 - \left(\frac{1}{i}\nabla - eA\right)^2\right]\Psi = m^2\Psi \qquad \text{...(3.1)}$$

The equation (3·1) represents a spinless particle of charge e and mass m in a scalar potential ϕ and a vector potential $A(r, t)$. The natural units ($\hbar = c = 1$) are used.

If we consider the vector potential $A = 0$ and assume the scalar potential ϕ to be time independent, (3·1) gives

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2} - 2ie\,\Phi\,\frac{\partial}{\partial t} + e^2\,\Phi^2\right]\Psi = m^2\Psi. \qquad ...(3\cdot 2)$$

Considering

$$\Psi(r, t) = u(r)\,e^{-iEt}, \qquad ...(3\cdot 3)$$

we obtain from (3·2)

$$[\nabla^2 + (E - e\Phi)^2]\,u(r) = m^2\,u(r). \qquad ...(3\cdot 4)$$

The eqn. (3·4) is the same as obtained by Schiff [1955, (42·13), p. 321].

If we separate the variables as

$$u(r, \theta, \phi) = R(r)\,Y_{lm}(\theta, \phi), \qquad ...(3\cdot 5)$$

from (3·4) we get the radial equation with the scalar potential for the Klein-Gordon equation $V_{kg} = e\Phi$,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[(E^2 - m^2) + (V_{kg} - 2E)\,V_{kg} - \frac{l(l+1)}{r^2}\right]R = 0$$

$$l = 0, 1, 2, \ldots \qquad ...(3\cdot 6)$$

Substituting $R = \chi_l/r$ in (3·6), we get

$$\frac{d^2\chi_l}{dr^2} + \left[(E^2 - m^2) + (V_{kg} - 2E)\,V_{kg} - \frac{l(l+1)}{r^2}\right]\chi_l = 0. \qquad ...(3\cdot 7)$$

For the s-wave ($l = 0$), eqn. (3·7) gives

$$\frac{d^2\chi_0}{dr^2} + [(E^2 - m^2) + (V_{kg} - 2E)\;V_{kg}]\;\chi_0 = 0. \qquad ...(3\cdot 8)$$

(II) *The Dirac Wave Equation*

The two coupled equations [Messiah 1970 (XX. 170 a, b), p. 928] which represents the radial Dirac wave equation for a spherically symmetric scalar potential V_d, are given as

$$\left[-\frac{d}{dr} + \frac{\omega(J+\frac{1}{2})}{r}\right]G = (E - m - V_d)\,F \qquad ...(3\cdot 9a)$$

$$\left[\frac{d}{dr} + \frac{\omega(J+\frac{1}{2})}{r}\right]F = (E + m - V_d)\,G \qquad ...(3\cdot 9b)$$

where F and G are the radial functions.

Following the suggestion of Feynman (1958) for a simple algebraic decoupling of (3.9a) and (3.9b), we consider them as simultaneous equations (Sengupta 1964), after eliminating G from (3.9a) with the help of (3.9b); we get

$$\frac{d^2F_l}{dr^2}+\left[(E-V_d)^2-m^2-\frac{l\,(l+1)}{r^2}\right]F_l=0 \qquad \text{...(3.10)}$$

where we have considered

$$(J+\tfrac{1}{2})(J+\omega+\tfrac{1}{2})=l\,(l+1). \qquad \text{...(3.11)}$$

Equation (3.10) can be rearranged to give

$$\frac{d^2F_l}{dr^2}+\left[(E^2-m^2)+(V_d-2E)\,V_d-\frac{l\,(l+1)}{r^2}\right]F_l=0. \qquad \text{...(3.10}a\text{)}$$

For the s-wave ($l=0$), we have

$$\frac{d^2F_0}{dr^2}+[(E^2-m^2)+(V_d-2E)\,V_d]\;F_0=0, \qquad \text{...(3.12)}$$

with a similar equation for the radial function G.

A comparison of eqns. (3.7) and (3.10a), and hence eqns. (3.8) and (3.12), shows that under the conditions for which the two coupled Dirac equations are considered as simultaneous equations (Sengupta 1964), the radial Dirac equation approaches the form of the radial Klein-Gordon equation. This is what can be expected from the correspondence principle (Messiah 1970, p. 890).

Also, if we consider

$$\chi_l=F_l=U_l \qquad \text{...(3.13)}$$

and write

$$V_{kg}=V_d=V_r \qquad \text{...(3.14)}$$

equations (3.8) and (3.12) can be represented by a single relativistic equation

$$\frac{d^2U_0}{dr^2}+[(E^2-m^2)+(V_r-2E)\,V_r]\,U_0=0. \qquad \text{...(3.15)}$$

We find that (3.15) is similar to the form (2.5a) of the non-relativistic Schrödinger equation and hence the general methods of Bhattacharjie and Sudarshan (1962) and Bose (1963), as discussed in section 2, can be used to construct solvable scalar potentials V_r.

4. Inter-relation Between Solvable Scalar Potentials

If a linear second order differential equation of the form (2.1) is transformed to the non-relativistic Schrödinger equation (2.5a); the necessary and sufficient conditions are given by (2·6*a*). However, if (2·1) be transformed to (3·15), the necessary and sufficient conditions are written as

$$A(r) = 0,\ B(r) = [(E^2 - m^2) + (V_r - 2E)\,V_r],\ \frac{\partial V_r}{\partial k} = 0 \qquad \text{...(4·1)}$$

Equating the values of $B(r)$ in (2·6 *a*) and (4·1), we have

$$(E^2 - m^2) + (V_r - 2E)\,V_r = 2m\,[(E - m) - V_s]. \qquad \text{...(4·2)}$$

From (4·2) we obtain an inter-relation between the solvable scalar potentials for the relativistic eqn. (3·15) and the non-relativistic Schrödinger eqn. (2·5*a*) as,

$$V_r = E \pm \{2m\,(E - V_s - \tfrac{1}{2}m)\}^{\frac{1}{2}}. \qquad \text{...(4·3)}$$

Following Schiff [1955, eqn. (42·21), p. 322], if we consider $E \approx m$, (4.2) gives

$$(V_r - 2E)\,V_r = -2mV_s. \qquad \text{...(4·4)}$$

In the units $\hbar = 2m = 1$ (4·4) becomes

$$2EV_r - V_r^2 = V_s, \qquad \text{...(4·4}a\text{)}$$

which is precisely the statement of Vasudevan *et al.* (1967), that the solvable potential V in the Schrödinger equation is replaced by $2EV - V^2$ in the radial Klein-Gordon equation to which our radial Dirac equation also reduces.

However, if we consider $E = m$ and $V_r \ll E$, (4·4) gives the non-relativistic limit, for which

$$V_r = V_s. \qquad \text{...(4·5)}$$

5. Discussion

The radial Klein-Gordon equation (3·8) is similar to the one used by Vasudevan *et al.* (1967, eqn. 33) with a substitution $E^2 = k^2 - m^2$. The radial Dirac equations (3·9), as they neglect spin interaction describe particles with zero spin similar to the Klein-Gordon equation. However, if we ignore the conservation of parity and treat the two coupled equations as simultaneous equations, they reduce exactly to the radial Klein-Gordon equation as can be expected from the corresponding principle. Thus, (3·15) represents a radial *s*-wave equation for a relativistic particle with zero spin and without a sharp parity. Solvable scalar potentials for this equation can be constructed by the methods of Bhattacharjie and Sudarshan (1962) and Bose (1963). The inter-relations between solvable scalar potentials for the two classes of equations are given by (4·3), (4·4) and (4·5) under the given approximations.

According to Case (1950), the Coulomb potential is strongly singular for the Dirac equation. Vasudevan *et al.*, (1967) observe that the r^{-4} potential for the *s*-wave Schrödinger equation is the same as r^{-2} potential for the corresponding Dirac equations. These statements are compatible with (4·3) and (4·4).

It is seen that in (4·3) as well as in (4·4) the energy E enters as a parametric term in the potential function. This fact can be viewed in the light of the properties of the velocity dependent potentials (Ferreira *et al.* 1967). A velocity dependent potential function for a Schrödinger equation can be replaced by a radial potential function [Ferreira *et al.* 1967, eqn. (3·6)] of the form

$$V_s(x) = -E\lambda M(x)/[1 - \lambda M(x)] \quad \ldots(5\cdot1)$$

where $M(x)$ is a real function of the Cartesian coordinate (for a one-dimensional Schrödinger equation) and λ is a dimensionless constant.

The energy dependence of the relativistic wave equation (3·15) can be thought of because of the velocity dependent potentials and the equality of V_s in (4·4) and (5·1) can be stipulated to give

$$\lambda M(x) = \left\{1 - \frac{2mE}{V_r(2E - V_r)}\right\}^{-1}, \quad \ldots(5\cdot2)$$

which has the poles for

$$V_r = E \pm \{E(E - 2m)\}^{\frac{1}{2}}. \quad \ldots(5\cdot3)$$

For $E = m$, also for which (4.4) is valid, (5·3) reduces to

$$V_r = E \pm iE, \quad \ldots(5\cdot4)$$

Equation (5·4) requires Vr, to be complex. Thus, $\lambda M(x)$ has poles only if V_r be complex.

Acknowledgement

The author is thankful to Prof. R. C, Varma for some useful suggestion.

References

Aly, H. H., and Spector, R. M. (1965). Some solvable potentials for the Schordinger equation. *Nuovo Cim.*, **38**, 149–52.

Bargmann, V. (1949). On the connection between phase shifts and scattering potentials. *Rev. mod. Phys.*, **21**, 488–93.

Bhattacharjie, A., and Sudarshan, E. C. G. (1962). A class of solvable potentials. *Nuovo Cim.*, **25**, 864–79.

Biedenharn, L. C. (1972). A two component alternative to Dirac equation. *Acta. Phys. Austr. Suppl.*, **11**, 837–46.

Bose, A. K. (1963). Solvable potentials. *Phys. Lett.*, **7**, 245–46.

Case, K. M. (1950). Singular potentials. *Phys. Rev.*, **80**, 797–806.

Evans, W. D. (1970). The Dirac equations with a spherically symmetric scalar potential. *Quart. J. Math. Oxford* (2), **21**, 89–99.

Ferreira, Erasmo, M., Guillen, Nilo and Sesma, Javilr (1967). Properties of velocity dependent potentials. *J. Math. Phys.*, **8**, 2243–49.

Feynman, R. P., and Gell-Mann, M. (1958). Theory of Fermi interaction. *Phys. Rev.*, **109**, 193.

Khan, I. A. (1973). Derivation of a few solvable potentials for the Schrödinger equation. *Indian J. pure appl. Math.*, **4**, 90–101.

Lemieux, A., and Bose, A. K. (1969). Construction of potentials for which the Schrödinger equation is soluble. *Ann. Inst. Poincare*, A **10**, 259–70 (in French).

Messiah, A. (1970). Quantum Mechanics, Vol. 2. North-Holland Publishing Co., Amsterdam, pp. 928–29.

Morse, P. M., and Feshbach, H. (1953). Methods of Theoretical Physics, Part II, McGraw-Hill Book Co., Inc., Tokyo. International student edition, pp. 1670–73.

Rajput, Kamal, S. (1976). A new solvable potential from the quasi-linear van der Pol equation. *Indian J. pure appl. Math.* (in press).

Schiff, L. I. (1955). Quantum Mechanics. McGraw-Hill Book Co., Inc., New York, Second edition, pp. 320–22.

Sengupta, N. D. (1947). On the exact solution of Dirac equation in the field of electromagnetic radiation. *Bull. Calcutta math. Soc.*, **39**, 147–53.

——— (1964). Exact solutions of the relativistic equation of motion of electron for a class of electromagnetic fields (classical and wave-mechanical). *S. N. Bose 70th Birth Day Commemoration Volume.* (S. N. Bose 70th Birth Day Celebration Committee, Calcutta), Part II, pp. 55–64.

Sharma, L. K. (1970*a*). Potentials from the modified Mathieu equation. *Proc. Indian natn. Sci. Acad.*, **36**, 230–38.

——— (1970*b*). Solvable potentials from associated Legendre equation. *Proc. Indian natn. Sci. Acad.*, **36**, 239–45.

Sharma, L. K., and Singh, F. (1971). Solvable potentials from the Sharpe's Hermite and Laguerre equations. *Indian J. Phys.*, **45**, 1–8.

Sharma, L. K., and Varma, R. C. (1970). Derivation of a few solvable potentials for the Schrödinger equation. *Indian J. pure appl. Phys.*, **8**, 66–69.

Soff, G., Muller, B., Johann, R., and Greiner, W. (1973). Solution of the Dirac equation for scalar potentials and its implication in atomic physics. *Z. Naturforsch.*, **28** *a*, 1389–96.

Stanciu, G. N. (1966). Solvable vector and scalar potentials for the Dirac equation. *Phys. Lett.*, **23**, 232–33.

Vasudevan, R., Venktesan, K., and Jaganathan, G. (1967). Construction of solvable potentials and some aspects of regularization of singular potentials. *Nuovo Cim. suppl.*, **5**, 621–41.