A SINGULAR PERTURBATION SOLUTION OF DOUBLE PHASE FLOW DUE TO DIFFERENTIAL WETTABILITY

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A singular perturbation solution of the phenomenon of imbibition which arises in the flow of two immiscible liquids through porous media due to the difference in the wetting abilities of liquids has been discussed here.

1. INTRODUCTION

We have analytically discussed here the phenomenon of imbibition in porous media (Scheidegger 1960), under certain conditions, by using a singular perturbation approach (Nayfeh 1973). This phenomenon arises due to the difference in the wetting abilities of the flowing phases and has been investigated by many authors (Graham and Richardson 1959; Verma 1969, 1972; Verma and Mishra 1974).

We consider that a finite cylindrical piece of homogeneous porous matrix of length L is fully saturated with a native liquid N. It is completely surrounded by an impermeable surface except for one end of the cylinder which is labelled as imbibition face, and this end is exposed to an adjacent formation of the injected liquid I. It is assumed that the liquid I is preferentially wetting phase and so this arrangement gives rise to the phenomenon of imbibition.

2. FORMULATION AND SOLUTION

Assuming that Darcy's law is valid in the case being investigated, we may write the basic flow equations governing imbibition phenomenon as

$$v_i = -\frac{k_i}{\delta_i} K \frac{\partial p_i}{\partial x} \qquad \text{...(2.1)}$$

$$v_n = -\frac{k_n}{\delta_n} K \frac{\partial p_n}{\partial x} \qquad \text{...(2.2)}$$

$$v_i = -v_n \qquad \text{...(2.3)}$$

$$p_c = p_n - p_i = f(S_i), \text{ say.} \qquad \text{...(2.4)}$$

$$\rho \frac{\partial S_i}{\partial t} + \frac{\partial v_i}{\partial x} = 0 \qquad \text{...(2.5)}$$

where v_i and v_n are the velocities, k_i and k_n the relative permeabilities, δ_i and δ_n the viscosities, p_i and p_n the pressures of the injected and native liquid respectively; ρ and K are the porosity and permeability of the medium; S_i is the saturation of injected liquid, p_c is capillary pressure, x is linear co-ordinate, and t is the time.

Combining eqns. (2.1) - (2.5) and using the relation for capillary pressure as $p_c = \beta S_w$ (Verma 1969), we get

$$\rho \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x}\left[KD(S_i)\beta \frac{\partial S_i}{\partial x}\right] = 0 \quad ...(2.6)$$

where

$$D(S_i) = \frac{k_i k_n}{(\delta_n k_i + \delta_i k_n)}, \text{ β being small capillary pressure coefficient.} \quad ...(2.7)$$

A set of boundary conditions may be written as

$$S_i(0, t) = S_{i0},\ S_i(L, t) = S_{il} \quad ...(2.8)$$

where S_{i0} and S_{il} are the saturations at the imbibition face and at the end $x = L$ respectively.

For further discussion we assume an average value of $D(S_i)$, say $\bar{D}(S_i)$; using the transformation

$$\xi = \frac{x}{L},\quad \theta = \frac{K}{\rho L^2} t \quad ...(2.9)$$

equations (2.6) and (2.8) become

$$\frac{\partial S_i}{\partial \theta} + \beta \bar{D}(S_i) \frac{\partial S_i}{\partial \xi} = 0 \quad ...(2.10)$$

and

$$S_i(0, \theta) = S_{i0},\ S_i(1, \theta) = S_{il}. \quad ...(2.11)$$

Equation (2.10) together with boundary condition (2.11) constitute the desired differential system.

To solve (2.10) we use Birkhof's technique of one parameter transformation (Hansen 1964). Let a group T_1 consisting of a set of transformations be

$$T_1 : \bar{\xi} = a^p \xi,\ \bar{\theta} = a^r \theta, \text{ and } \bar{S}_i = a^s S_i \quad ...(2.12)$$

where $a \neq 0$ and p, r, s are real numbers to be determined.

Substituting these values in equation (2.10) and applying condition of absolute conformal invariant under T_1, the invariant of the group T_1 are given by

$$\eta = \frac{\xi}{\sqrt{\theta}},\ F(\eta) = \frac{S_i(\xi, \theta)}{\theta^A} \qquad \ldots(2.13)$$

where A is the free parameter.

Plugging in these values in (2.10) and substituting

$$F(\eta) = u(z),\ 2z = -\alpha\eta^2 \ \ (\text{where } \alpha = 1/2\bar{D}) \qquad \ldots(2.14)$$

we get

$$\beta u''(z) + a(z)\,u'(z) + b(z)\,u(z) = 0,\ u(0) = S_{i_0},\ u\left(-\frac{\alpha}{2\theta}\right) = S_{il} \qquad \ldots(2.15)$$

where

$$a(z) = \frac{1}{z}\left(\frac{\beta}{2} - z\right),\ b(z) = -\frac{A}{z}.$$

By setting $\beta = 0$ and using second boundary condition of (2.15), we obtain the outer expansion (valid near $x = L$) as

$$u^0 = S_{il} \exp\left[-\int_{-\alpha/2\theta}^{z} \frac{b(v)}{a(v)}\,dv\right]. \qquad \ldots(2.16)$$

It may be mentioned that the first boundary condition of (2.15) has been dropped because $a(z) > 0$ in $\left(0, -\frac{\alpha}{2\theta}\right)$, (cf. Nayfeh 1973).

Using the stretching transformation $y = \frac{z}{\beta}$ and the first boundary condition of (2.15), we get the inner expansion, valid near origin (Nayfeh 1973) as

$$u' = S_{i_0} - B + Be^{a(0)y}. \qquad \ldots(2.17)$$

Applying matching principle to u' and u^0, evaluating certain integrals, forming the composite expansion (Nayfeh 1973) and using (2.13) and (2.14), we finally get

$$S_i^c(\xi, \theta) = \theta^A \left(\frac{\frac{\beta}{2} - \frac{1}{4\bar{D}\theta}}{\frac{\beta}{2} - \frac{\xi}{4\bar{D}\theta}}\right)^{-A} + \theta^A \left\{ S_{i_0} - S_{il}\left(\frac{\frac{\beta}{2} - \frac{1}{4\bar{D}\theta}}{\frac{\beta}{2}}\right)^{-A} \right\}$$

$$\times \exp\left(\tfrac{1}{2} + \frac{\xi}{4\bar{D}\theta}\right) + O(\beta). \qquad \ldots(2.18)$$

This is the required saturation distribution of the displacing liquid in terms of the saturation at the two ends of the porous matrix.

We have thus obtained a uniformly valid composite solution describing the displacing phase saturation by applying the matching principle of the inner and outer expansions.

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