WAVES IN A HEAVY INCOMPRESSIBLE FLUID OF FINITE DEPTH AND OF VARIABLE DENSITY

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The character of equilibrium of heavy, viscous, incompressible, finitely conducting fluid of variable density in the presence of a magnetic field along the direction of gravitational field has been investigated, when the lower bounding surface is rigid and the upper is free. Based on the existence of variational principle which characterizes the solution, approximate solution has been derived for the case of a fluid having exponentially varying density in the vertical direction. The nature of the boundaries chosen gives rise to an additional monotonically decreasing mode compared to the cases where the boundaries chosen are both free or both rigid.

1. INTRODUCTION

The character of an incompressible, inviscid fluid of variable density stratified in the vertical direction was first investigated by Rayleigh (1883) and he found that the configuration is stable or unstable according as $\frac{d\rho_0}{dz}$ is everywhere negative or anywhere positive. Chandrasekhar (1955) studied the effects of viscosity on the same problem.

Hide (1955) considered the problem of waves in a heavy compressible viscous electrically conducting fiuid in the presence of a magnetic field which is uniform and is directed in a direction parallel to the force of gravity. He obtained the solution in terms of integrals and showed that it is characterized by a variational principle. Based on the existence of this variational principle he further obtained the dispersion relation for a fluid having exponentially varying density and confined between two free boundaries.

Ariel and Bhatia (1966) studied the same problem, where the fluid is confined between two boundaries which are rigid and ideal conductors, and showed that the solution is similar to that obtained by Hide.

The present note investigates theoretically the same problem as considered by Hide when the fluid is confined between two boundaries, the lower bounding surface being rigid and the upper free. The explicit solutions have been carried out for the following two cases :

(1) Waves in an ideal conductor in the presence of buoyancy forces,

(2) Waves in the absence of buoyancy forces.

2. Formulation of the Problem

The equation of motion suitable for variational procedure is [Hide 1955, equation (47)]

$$n_i \int_L \rho_0(k^2 w_i w_j + Dw_i Dw_j)\, dz - \frac{gk^2}{n_i} \int_L D\rho_0 w_i w_j\, dz + \int_L \mu_0(k^4 w_i w_j$$

$$+ 2k^2 Dw_i Dw_j + D^2 w_i D^2 w_j)\, dz + k^2 \int_L D^2\mu_0 w_i w_j\, dz$$

$$+ \frac{\kappa k^2}{4\pi}(n_j + \eta k^2) \int_L h_i h_j\, dz + \frac{k}{4\pi}(n_j + 2\eta k^2) \int_L Dh_i Dh_j\, dz$$

$$+ \frac{\kappa\eta}{4\pi k^2} \int_L D^2 h_i D^2 h_j\, dz = 0, \qquad \text{...(1)}$$

where ρ_0 denotes the density before the system is disturbed, w the z-component of the velocity of perturbation, g the acceleration due to gravity $(0, 0, -g)$, μ the coefficient of viscosity, H_0 the magnetic field directed towards the z-axis in the undisturbed state, κ the coefficient of magnetic permeability, h the z-component of the magnetic field in the perturbed state, n growth rate of disturbance, k wave number, components $(k_x, k_y, 0)$, η the coefficient of electrical conductivity, L the vertical extent of the fluid, and D the differentiation with respect to z.

Further h and w are connected by the relation

$$[n - \eta(D^2 - k^2)]\, h = H_0 Dw. \qquad \text{...(2)}$$

3. Boundary Conditions

The fluid is assumed to be confined to the planes $z = 0$ which is rigid and perfectly conducting and $z = d$ which is free boundary.

The appropriate boundary conditions for the present problem are

$$\text{(i)}\quad w(0) = Dw(0) = h(0) = 0, \qquad \text{...(3)}$$

$$\text{(ii)}\quad w(d) = D^2w(d) = Dh(0) = 0. \qquad \text{...(4)}$$

4. The Case of Exponentially Varying Density

We consider the problem of a continuously stratified fluid of depth d in which the undisturbed density distribution is given by

$$\rho_0(z) = \rho_1 \exp \beta z \qquad \text{...(5)}$$

where ρ_1 and β are constants.

It would be convenient to assume

$$\mu_0(z) = \nu\rho_1 \exp z \qquad \text{...(6)}$$

where ν is the coefficient of kinematic viscosity.

In order to ensure that the density variation within the fluid is small compared to the average density, we make an assumption

$$| \beta d | < < 1. \qquad \text{...(7)}$$

Let us assume the following trial function for $w(z)$ which satisfies the boundary conditions (3) and (4)

$$w(z) = W(\cos lz - \cos 3lz), \qquad \text{...(8)}$$

where $l = \pi s/2d$, s being an odd integer.

The value of $h(z)$ can be directly obtained by substituting the value of $w(z)$ in eqn. (2), we have

$$h(z) = lH_0W\left[\frac{3 \sin 3lz}{n + \eta(9l^2 + k^2)} - \frac{\sin lz}{n + \eta(l^2 + k^2)}\right]. \qquad \text{...(9)}$$

Evaluating the integrals $(i = j)$ of eqn. (1) by substituting the values of w and h, we obtain

$$y^2 + \frac{S}{2}\left(\frac{16x^4 + 40x^2 + 41}{4x^2 + 5}\right)y - \frac{4x^2B}{4x^2 + 5} + \frac{y}{2}\left[\frac{4x^2 + 1}{2y + R(4x^2 + 1)}\right.$$
$$\left. + \frac{9(4x^2 + 9)}{2y + R(4x^2 + 9)}\right] = 0 \qquad \text{...(10)}$$

where

$$x = kd/\pi s$$

$$y = (nd/\pi s)\,(4\pi\rho_1/\kappa H_0^2)^{1/2}$$

$$R = \eta\pi s/2dV_0$$

$$B = g\beta d^2/\pi^2 s^2 V_0^2$$

$$S = \nu\pi s/2dV_0$$

and

$$V_0^2 = \kappa H_0^2/4\pi\rho_1. \qquad \text{...(11)}$$

Thus to specify y the dimensionless growth rate for any given x the dimensionless wave number, we require three parameters R, B and S. They are respectively the suitable measures of finite conductivity, buoyancy forces and viscous forces in terms of magnetic field.

Equation (10) is biquadratic in y, but reduces to a quadratic equation in two cases, i.e. when R is equal to zero and R is equal to infinity. Equation (10) reduces to cubic in case of $B = 0$.

It may be recalled here that the corresponding dispersion relation obtained by Hide (1955) and Ariel and Bhatia (1966) for free and rigid boundaries were cubics. Thus in our present problem there is an additional mode which, however, decays exponentially with time. This mode has been discussed in detail for the case $B = 0$. (The restriction $B = 0$ does not alter its character).

5. Case I — Waves in an Ideal Conductor in the Presence of Buoyancy Forces

Substituting $R = 0$ in equation (10), we get

$$y^2 + \frac{S}{2}\left(\frac{16x^4 + 40x^2 + 41}{4x^2 + 5}\right) y - \frac{4Bx^2}{4x^2 + 5} + \frac{20x^2 + 41}{2(4x^2 + 5)} = 0. \quad \text{...(12)}$$

Its solution is

$$y = -\frac{S}{2}\left(\frac{16x^4 + 40x^2 + 41}{4x^2 + 5}\right) \pm \left[\frac{S^2}{4}\left(\frac{16x^4 + 40x^2 + 41}{4x^2 + 5}\right) - \left(\frac{40x^2 - 16Bx^2 + 82}{4x^2 + 5}\right)\right]^{1/2} \quad \text{...(13)}$$

(i) *Unstable Stratification* $\beta > 0$, $B > 0$.

When $x > x_0$, x_0 being the real positive root of the equation

$$4(2B - 5)\, x_0^2 = 41 \quad \text{...(14)}$$

the value of y corresponding to the upper sign is real and positive. In these circumstances the disturbance grows aperiodically with time, therefore, the equilibrium is unstable. There exists one value of x(x_m say) for which the growth rate is maximum and equal to y_m (say). The expressions for y_m and x_m are given by

$$2y_m^2\,(4x_m^2 + 5) + S(16x_m^4 + 40x_m^2 + 41) + x_m^2\,(20 - 8B) + 41 = 0 \quad \text{...(15)}$$

and

$$2y_m^2 + 2S(4x_m^2 + 5)\, y_m + (5 - 2B) = 0. \quad \text{...(16)}$$

When $x < x_0$, in eqn. (14), y no longer has a positive value, so that for these values of x, no amplified motion is possible. The two remaining possibilities are aperiodically or periodically damped flow. However, such motion would not be observed because unstable motion at $x > x_0$ would swamp any other type of motion — the mathematically stable modes of this case of unstable stratification have no physical interest.

(ii) *Stable Stratification* : $\beta < 0$, $B < 0$.

When $\beta < 0$, $B < 0$ and by eqn. (13)

$$y = -\frac{S}{2}\left(\frac{16x^4 + 40x^2 + 41}{4x^2 + 5}\right)y \pm \left[\frac{S^2}{4}\left(\frac{16x^4 + 40x^2 + 41}{4x^2 + 5}\right)^2 - \left(\frac{16B_0x^2 + 40x^2 + 82}{4x^2 + 5}\right)\right]^{1/2}, \; (B = -B_0) \qquad \text{...(17)}$$

from which it follows that y never has a positive real part so that the equilibrium is stable. Whether it is restored periodically or aperiodically depends on the sign of the quantity under the radical sign.

It will be convenient to introduce the quantity

$$Z = \frac{4(4x^2 + 5)(40x^2 + 16B_0x^2 + 82)}{(16x^4 + 40x^2 + 41)^2}. \qquad \text{...(18)}$$

It follows from eqn. (18) that the motion is periodically or aperiodically damped according as $S^2 \lesseqgtr Z$. We have only oscillations at those values of x for which a line drawn parallel to the x-axis at a distance S^2 above it lies below the curve Z; otherwise the equilibrium is restored aperiodically. In Fig. 1, Z is plotted as a function of x for different values of B_0. We note that Z achieves its maximum value Z_m at $x = x_m$ given by equations

$$64x_m^6(2B_0 + 5) + 48x_m^4(5B_0 + 33) + 4x_m^2(535 - 32B_0) - 205(B_0 - 3.4) = 0 \qquad \text{...(19)}$$

and

$$Z_m = \frac{8(4x_m^2 + 5)(20x_m^2 + 8B_0x_m^2 + 41)}{(16x_m^4 + 40x_m^2 + 41)^2} \qquad \text{...(20)}$$

It becomes clear from eqn. (19) that if $B_0 > 3.4$, there will be just one value of x_m satisfying it. On the other hand if $B_0 \leqslant 3.4$, no value of x_m is admissible as can be seen from Descartes rules of sign.

Thus in Fig. 1, we must distinguish the two cases namely $B_0 \leqslant 3.4$ and $B_0 > 3.4$. When $B_0 \leqslant 3.4$, Z falls monotonically with x from its value (40/41) at $x = 0$ approaching the x-axis asymptotically, when $B_0 > 3.4$, Z rises from the value at $x = 0$ until it reaches Z_m and then again tails off asymptotically to x-axis.

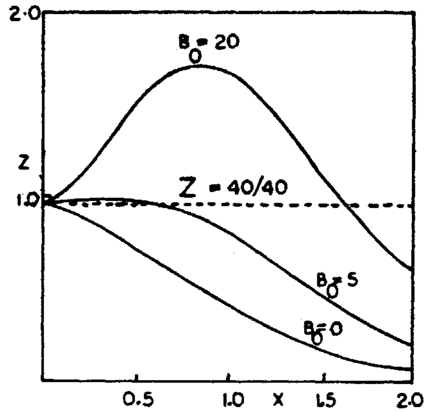

FIG. 1. Variation of Z with x for different values of B_0 ($B_0 = -B$).

The situation may now be summarized as follows :

(a) $B_0 \leqslant 3.4$, $S^2 \geqslant 40/41$. In this case $S^2 \geqslant Z$ for all values of x, so that all modes are aperiodically damped.

(b) $B_0 \leqslant 3.4$, $S^2 < 40/41$. There is one value of x, which will be termed x_1 for which $S^2 = Z$. Oscillations arise when $0 < x < x_1$, and aperiodic motion when $x \geqslant x_1$.

(c) $B_0 > 3.4$, $S^2 \geqslant Z_m$. Again, in this case $S^2 > Z$ for all values of x, so that all modes are aperiodically damped.

(d) $B_0 \geqslant 3.4$, $(40/41) < S^2 < Z_m$. There are two non-zero values of x, which will be termed x_2 and x_3, for which $S^2 = Z$. Oscillations arise only within the wave number range $x_2 < x <_3$, and aperiodic motion takes place elsewhere.

In the following table the value of x_m and Z_m are listed for the different values of B_0.

S. No.	B_0	x_m	Z_m
1	4	0.2610	0.9793
2	6	0.4947	1.0296
3	8	0.6048	1.1229
4	10	0.6726	1.2076
5	12	0.7191	1.3087
6	14	0.7531	1.4220
7	16	0.7790	1.5373
8	18	0.7995	1.6495
9	20	0.8161	1.7023

It remains now to specify the properties of the motion. In the case of aperiodic damping there are two damping coefficients [see eqn. (21)].

$$-y = \frac{S}{2}\left(\frac{16x^4+40x^2+41}{4x^2+5}\right) \pm \left[\frac{S^2}{2}\left(\frac{16x^4+40x^2+41}{4x^2+5}\right)^2 - \left(\frac{16B_0x^2+40x^2+82}{4x^2+5}\right)\right]^{1/2}. \quad \text{...(21)}$$

In the case of oscillatory motion, there is only one damping coefficient,

$$-R(y) = \frac{S}{2}\left(\frac{16x^4+40x^2+41}{4x^2+5}\right). \quad \text{...(22)}$$

The angular frequency of the oscillation is given by

$$I(y) = \pm\left[\left(\frac{16B_0x^2+40x^2+82}{4x^2+5}\right) - \frac{S^2}{4}\left(\frac{16x^4+40x^2+41}{4x^2+5}\right)^2\right]^{1/2}. \quad \text{...(23)}$$

These functions are illustrated by Figs. 2, 3 and 4 for cases (b), (d) and (c) respectively.

The corresponding wave and group velocities V_w and V_g satisfy the equations

$$V_w = I(y)/x \quad \text{...(24)}$$

$$V_g = dI(y)/dx \quad \text{...(25)}$$

$$= \frac{2}{V_w}\left[\frac{10(5B_0-8)}{(4x^2+5)^2} - \frac{S^2(16x^4+40x^2+9)(16x^4+40x^2+41)}{(4x^2+5)^3}\right]. \quad \text{...(26)}$$

These functions are illustrated by Figs. 5 and 6.

6. Case II — Waves in the Absence of Buoyancy Forces

Substituting $B = 0$, in eqn. (10), we obtain the following cubic

$$8y^3(4x^2+5) + 4y^2[2R(4x^2+5)^2 + S(4x^2+5)^2 + 16] + 2y[(4x^2+5)\{R^2((4x^2+5)^2-16) + 2RS((4x^2+5)^2+16) + 10\} + 32] + R^2S((4x^2+5)^4-256) + 10R((4x^2+5)^2-16) = 0. \quad \text{...(27)}$$

It is worth mentioning here that for the other type of boundaries (both free — Hide 1955 a, b; both rigid — Ariel and Bhatia 1966) the corresponding dispersion relation is quadratic. Thus we see that the additional mode is preserved in this case.

We shall now examine the behaviour of this mode in detail. Since both R and S are positive, y never has a positive real part. Hence the equilibrium is thoroughly

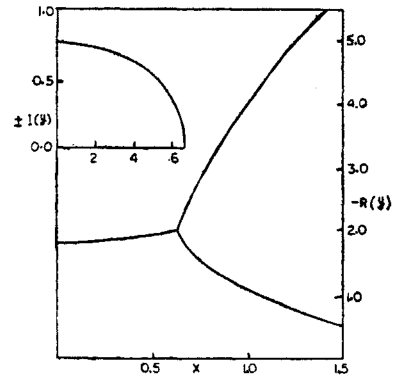

FIG. 2. The growth rate y, as a function of the wavenumber x for $R = 0$, $S = \sqrt{0.8}$ and $B_0 = 1.6$. In this case waves arise within the range $0 < x < x_1$, where $x_1 = 0.64$.

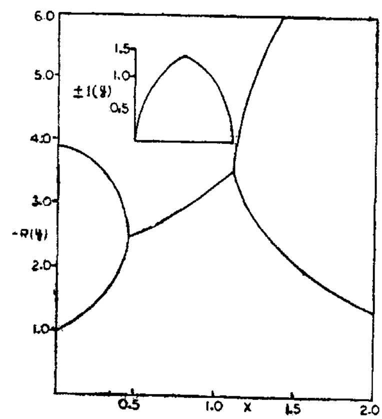

FIG. 3. The growth rate y, as a function of the wavenumber x for $R = 0$, $S = \sqrt{1.5}$ and $B_0 = 20$. In this case waves arise within the range $x_2 < x < x_3$, where $x_2 = 0.479$ and $x_3 = 1.19$.

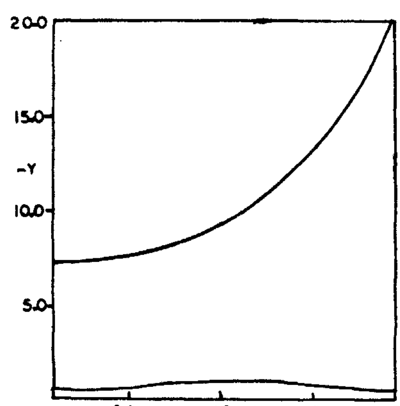

FIG. 4. The damping coefficient $-y$, as a function of the wavenumber x in case $R = 0$, $S = 2.0$ and $B_0 = 10$.

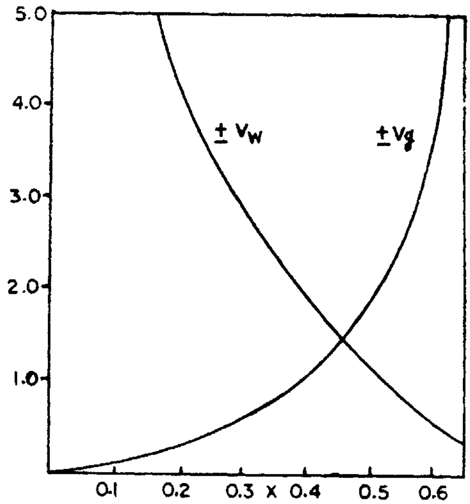

FIG. 5. The phase and group velocities V_w and V_g respectively as a function of the wavenumber x in case $R = 0$, $S = \sqrt{0.8}$ and $B_0 = 1.6$.

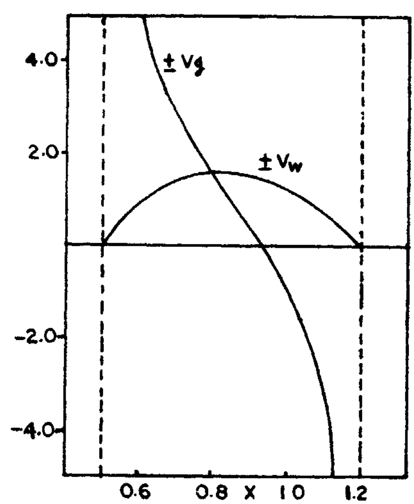

FIG. 6. The phase and group velocities V_w and V_g respectively, as a function of the wavenumber x in case $R = 0$, $S = \sqrt{1.5}$ and $B_0 = 20$.

stable and motion is damped. Equation (27) being cubic, admits either three real negative roots or two complex conjugate roots with negative real parts and one negative root. In the later case the two modes would give rise to oscillatory motion. It is extremely difficult to find the general conditions in which such possibility would take place. However, by keeping R fixed (say 0.2) and varying S we can draw a few general conclusions.

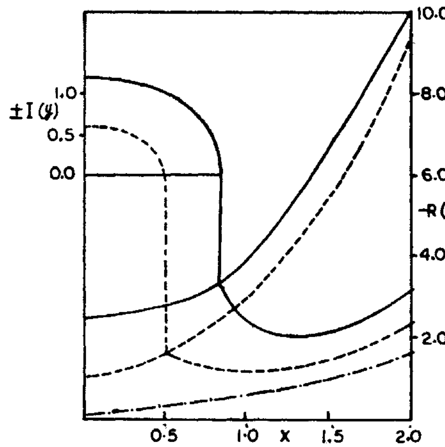

FIG. 7. The decay rate of y, as a function of wavenumber x for $R = 0.2$ and $S = 1.0$. In case $B = 0$. —— represent the decay rate for one rigid and one free boundary, and ----- represent the decay rate for the free boundaries, — · — · — · — · — represent the additional mode.

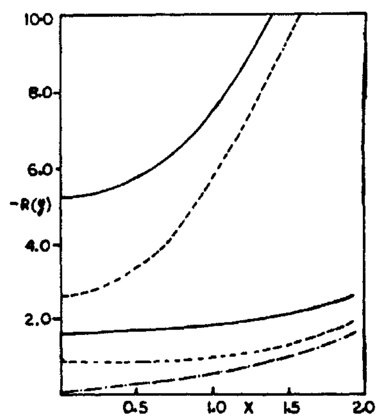

FIG. 8. The decay rate of y as a function of x for $R = 0.2$ and $S = 1.5$ in case $B = 0$. —— represent the decay rate for one rigid and one free boundary, and ----- represent the decay rate for free boundaries — · — · — · — · — represent the additional mode.

A careful study of eqn. (27) reveals that there exists a value $S^*(R)$ depending on R such that for $S < S^*(R)$, two roots of equation (27) are complex conjugate near $x = 0$, and for $S > S^*(R)$ no root is complex for any value of x. This value of $S^*(R)$ of S can be obtained by applying the condition of equality of two roots of eqn. (27) at $x = 0$ and for $R = 0.2$, $S^*(0.2) = 1.206$. Thus it becomes clear that we must distinguish between the following two cases.

(i) $S < S^*(R)$. In this case we note that motion takes place periodically for small values of x and aperiodically for large values of x. The oscillation arises for the range $0 < x < x_i$. The motion is aperiodic elsewhere. Fig. 7 gives the variation of y against x for $R = 0.2$ and $S = 1.0$ according to equation (27).

(ii) $S > S^*(R)$. In this case the motion is aperiodically damped for all values of x. Fig. 8 gives the variation of y against x for $R = 0.2$ and $S = 1.5$ according to eqn. (27).

A comparison has also been made with the corresponding growth rate of decay for two free boundaries. It can also be seen from Figs. 7 and 8 that the particular nature of the boundaries chosen by us gives rise to an additional monotonically decreasing mode.

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