ON | C, 1 | - SUMMABILITY OF JACOBI SERIES

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Let Σa_n be an infinite series with partial sums S_n. If $\sum \frac{|S_n|}{n^\delta} < \infty, \delta > 0$ then we observe that Σa_n is summable $| C, \delta |$. With the help of this result we prove that the Jacobi series at end points of the interval $[-1, +1]$ is summable $| C, 1 |$; provided that the generating function belongs to a certain Lipschitz class.

§1. Let Σu_n be an infinite series whose n-th partial sum is S_n. Let S_n^α and t_n^α be nth Cesàro mean of order α of the sequences $\{S_n\}$ and $\{nu_n\}$ respectively. The series Σu_n is said to be summable $| C, \alpha |$, $\alpha > -1$ or absolutely summable (C, α) if

$$\Sigma \, | S_n^\alpha - S_{n-1}^\alpha | < \infty$$

or

$$\Sigma \, n^{-1} \, | t_n^\alpha | < \infty. \qquad \ldots(1.1)$$

§2. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha > -1$, $\beta > -1$ are defined by the following expansion:

$$2^{\alpha+\beta} (1 - 2xt + t^2)^{-(1/2)} [1 - t + (1 - 2xt + t^2)^{1/2}]^{-\alpha}$$
$$\times [1 + t + (1 - 2xt + t^2)^{1/2}]^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) \, t^n. \qquad \ldots(2.1)$$

Let $f(x)$ be a function defined on the linear interval $-1 \leqslant x \leqslant +1$, such that the integral

$$\int_{-1}^{+1} (1 - x)^\alpha (1 + x)^\beta f(x) \, dx$$

exists in the sense of Lebesgue. The Fourier-Jacobi expansion generally known as Jacobi series corresponding to the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \qquad \ldots(2.2)$$

where

$$a_n = \frac{1}{g_n} \int_{-1}^{+1} (1-t)^{\alpha} (1+t)^{\beta} f(t)\, P_n^{(\alpha,\beta)}(t)\, dt$$

and
$$g_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1)\,.\,\Gamma(n+\beta+1)}{\Gamma(n+1)\,.\,\Gamma(n+\alpha+\beta+1)}.$$

Ultraspherical and Legendre series are particular cases of the series (2.2) when $\alpha = \beta = \lambda - \frac{1}{2}$ and $\alpha = \beta = 0$ respectively.

We write

$$\phi(w) = [f(\cos w) - A]$$

where A is a fixed constant. Recently Pandey (1968) has proved the following theorem in case of ordinary Cesàro summability.

Theorem — The series (2.2) for $x = \cos\theta$ is summable (C, k), for $\alpha - \frac{1}{2} < k < \alpha + \frac{1}{2}$, $-\frac{1}{2} < \alpha < \frac{1}{2}$ at $\theta = 0$ to the sum A provided that

$$\phi(w) \in \text{lip}^* \,(\alpha + \tfrac{1}{2} - k) \qquad \ldots(2.3)$$

where $\quad f(x) \in \text{lip}^*_{\alpha}$ means $| f(x+u) - f(x) | = o(u^{\alpha})$; $\alpha > 0$.

It is known that ordinary convergence together with absolute Abel summability of an infinite series do not necessarily imply the $| C, 1 |$ - summability of that series. We prove $| C, 1 |$ - summability of the series (2.2) under a condition of Lipschitz class. In the next section relevant results about asymptotic orders of Jacobi polynomials along with a result about infinite series have been given.

§3. Following results are used to prove the main theorem of this paper.

Lemma 1 — Let $\Sigma\, u_n$ be an infinite series with nth partial sum S_n.

If

$$\sum_{k=1}^{\infty} \frac{| S_k |}{k^{\delta}} < \infty \qquad \ldots(3.1)$$

then the series $\Sigma\, u_n$ is summable $| C, \delta |$; $0 < \delta \leqslant 1$.

Proof : Let t_n^{δ} and S_n^{δ} be the nth Cesàro mean of order δ of the sequences $\{nu_n\}$ and $\{S_n\}$ respectively, then we have

$$t_n^{\delta} = \delta(S_n^{\delta-1} - S_n^{\delta})$$

(*equation continued on p.* 540)

$$= \delta \left\{ \frac{1}{A_n^{\delta-1}} \sum_{k=0}^{n} A_{n-k}^{\delta-2} S_k - \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} S_k \right\},$$

where $$A_n^{\delta} = \binom{n+\delta}{n} = \frac{(\delta+1)(\delta+2)\dots(\delta+n)}{n!} \simeq \frac{n^{\delta}}{\Gamma(\delta+1)};$$

$$\delta \neq -1, -2 \dots.$$

We consider the case $0 < \delta < 1$ while the case $\delta = 1$ is known (Bhatt 1967). Now, we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\delta} \mid \leqslant \left[\sum_{n=1}^{\infty} \frac{1}{nA_n^{\delta-1}} \left(\sum_{k=0}^{n} \mid A_{n-k}^{\delta-2} \mid . \mid S_k \mid \right) + \sum_{n=1}^{\infty} \frac{1}{nA_n^{\delta}} \left(\sum_{k=0}^{n} A_{n-k}^{\delta-1} \mid S_k \mid \right) \right]$$

$$= A + B, \text{ say.}$$

But

$$A \leqslant \sum_{k=0}^{\infty} \mid S_k \mid \left(\sum_{n=k+1}^{\infty} \frac{1}{n.A_n^{\delta-1}} \mid A_{n-k}^{\delta-2} \mid \right)$$

$$\leqslant \sum_{k=0}^{\infty} \frac{\mid S_k \mid}{(k+1)^{\delta}} \left(\sum_{n=k+1}^{\infty} \mid A_{n-k}^{\delta-2} \mid \right)$$

$$= O(1) . O(1)$$

and

$$B \leqslant \sum_{k=0}^{\infty} \mid S_k \mid . \left(\sum_{n=k+1}^{\infty} \frac{1}{nA_n^{\delta}} . A_{n-k}^{\delta-1} \right)$$

$$\leqslant \sum_{k=0}^{\infty} \frac{\mid S_k \mid}{(k+1)^{\delta}} \left(\sum_{n=k+1}^{\infty} \frac{1}{n(n-k)^{1-\delta}} \right)$$

$$= O(1).$$

This completes the proof of the Lemma.

Lemma 2 (Szegö 1967) — Let α, β be arbitrary and real, and C a fixed positive constant, $n \to \infty$. Then

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-(1/2)}.\ O(n^{-(1/2)}), \dfrac{C}{n} \leqslant \theta \leqslant \dfrac{\pi}{2} \\ O(n^{\alpha})0, \leqslant \theta \leqslant \dfrac{C}{n}. \end{cases} \qquad ...(3.2)$$

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-(1/2)} K(\theta) \{\cos(N\theta + \gamma) + (n\sin\theta)^{-1}.\ O(1)\}$$

$$; \frac{C}{n} \leqslant \theta \leqslant \pi - \frac{C}{n} \qquad ...(3.3)$$

where

$$K(\theta) = \pi^{-(1/2)} \left(\sin\frac{\theta}{2}\right)^{-\alpha-(1/2)} \left(\cos\frac{\theta}{2}\right)^{-\beta-(1/2)}$$

$$N = n + \tfrac{1}{2}(\alpha + \beta + 1), \gamma = -(\alpha + \tfrac{1}{2})\frac{\pi}{2}$$

§4. We prove the following:

Theorem — The series (2.2) for $x = \cos\theta$ is summable $|C, 1|$ at the point $\theta = 0$ provided that

$$\phi(w) \in \text{lip}\,\delta; (\delta > \alpha + \tfrac{1}{2}), -\tfrac{1}{2} < \alpha < \tfrac{1}{2}; \beta > -1, \qquad ...(4.1)$$

and the antipole condition that the integral $\int_0^t w^{\beta-(1/2)} |\phi(\pi - w)| \, dw < \infty \ (t \to 0)$ is satisfied. If $\beta \geqslant \alpha$ no antipole condition is required.

Proof : Let S_n be the nth partial sum of the series (2.2), hence we have,

$$S_n = \int_0^{\pi} (1 - \cos w)^{\alpha+(1/2)} (1 + \cos w)^{\beta+(1/2)} f(\cos w)$$

$$\times \sum_{m=0}^{n} \frac{1}{g_m} P_m^{(\alpha,\beta)}(\cos w)\, P_m^{(\alpha,\beta)}(1)\, dw.$$

$$S_n - A = \int_0^{\pi} \left(\sin\frac{w}{2}\right)^{2\alpha+1} \left(\cos\frac{w}{2}\right)^{2\beta+1} \phi(w). \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1).\Gamma(n + \beta + 1)}$$

$$\times P_n^{(\alpha+1,\beta)}.(\cos w)\, dw.$$

[by Szegö (1967, p. 71) and orthogonal property of Jacobi polynomials.]

We can set $A = f(1) = 0$, and consequently, we have

$$S_n = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1).\Gamma(n+\beta+1)} \int_0^{\pi} \left(\sin\frac{w}{2}\right)^{2\alpha+1}\left(\cos\frac{w}{2}\right)^{2\beta+1}\phi(w)$$

$$P_n^{(\alpha+1,\beta)}(\cos w)\,dw.$$

$$= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1).\Gamma(n+\beta+1)}\left\{\int_0^{\gamma_n} + \int_{\gamma_n}^{\pi-(1/n)} + \int_{\pi-(1/n)}^{\pi}\right\}$$

$$= J_1 + J_2 + J_3, \text{ say } \left(\gamma_n = \frac{C}{n}, C \text{ a fixed constant}\right).$$

$$J_1 = O(n^{2\alpha+2}) \int_0^{\gamma_n} w^{2\alpha+1+\delta}\,dw = O(n^{-\delta}). \qquad \ldots(4.2)$$

and

$$J_3 = O(n^{\alpha+\beta+1}) \int_0^{1/n} |\phi(\pi - w)| \,.\, w^{2\beta+1}\,dw$$

$$= O(n^{\alpha+\beta+1-2\beta-2}) \quad \text{or} \quad = O(n^{\alpha-(1/2)}) \text{ (by antipole condition)}$$

$$= (n^{\alpha-\beta-1}) \quad \text{or} = O(n^{\alpha-(1/2)}). \qquad \ldots(4.3)$$

Again

$$J_2 = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1).\Gamma(n+\beta+1)} \cdot \frac{n^{-(1/2)}}{\pi^{1/2}} \int_{\gamma_n}^{\pi-(1/n)} \left(\sin\frac{w}{2}\right)^{\alpha-(1/2)}\left(\cos\frac{w}{2}\right)^{\beta+(1/2)}$$

$$\times\ \phi(w)\,.\cos(Nw+\gamma)\,dw$$

$$+\ O(n^{\alpha-(1/2)}) \int_{\gamma_n}^{\pi-(1/n)} \left|\left(\sin\frac{w}{2}\right)^{\alpha-(3/2)}\left(\cos\frac{w}{2}\right)^{\beta-(1/2)}\phi(w)\right| dw$$

$$;\left(N = k + \frac{\alpha+\beta}{2} + 1\right)$$

$$= J_{2\cdot1} + J_{2\cdot2}, \text{ say.}$$

Now

$$J_{2\cdot2} = O(n^{\alpha-(1/2)}) \int_{\gamma_n}^{a} \left|\left(\sin\frac{w}{2}\right)^{\alpha-(3/2)}\left(\cos\frac{w}{2}\right)^{\beta-(1/2)}\phi(w)\right| dw +$$

(equation continued on p. 543)

$$+ O(n^{\alpha-(1/2)}) \int_{a}^{\pi-(1/n)} \left| \left(\sin \frac{w}{2}\right)^{\alpha-(3/2)} \left(\cos \frac{w}{2}\right)^{\beta-(1/2)} \phi(w) \right| dw$$

(a, a fixed constant)

$$= O(n^{\alpha-(1/2)}) \int_{\gamma_n}^{a} w^{\alpha-(3/2)+\delta} \, dw + O(u^{\alpha-(1/2)})$$

$$= O(n^{\alpha-(1/2)}) \left[w^{\alpha-(1/2)+\delta}\right]_{\gamma_n}^{a} + O(n^{\alpha-(1/2)})$$

$$= \begin{cases} O(n^{\alpha-(1/2)}), & \text{if } \alpha - \frac{1}{2} + \delta > 0 \\ O(n^{\alpha-(1/2)} \log n), & \text{if } \alpha - \frac{1}{2} + \delta = 0 \\ O(n^{-\delta}) & \text{if } \alpha - \frac{1}{2} + \delta < 0. \end{cases} \quad \ldots(4.4)$$

In the second integral antipole condition may be used if β is not greater or equal to α.

The integral in $J_{2\cdot 1}$ is the difference of real and imaginary parts of E, multiplied with $\cos \gamma$ and $\sin \gamma$ respectively, where

$$E = \int_{\gamma_n}^{\pi-(1/n)} \left(\sin \frac{w}{2}\right)^{\alpha-(1/2)} \left(\cos \frac{w}{2}\right)^{\beta+(1/2)} \phi(w) \, e^{iw(2n+\alpha+\beta+2)/2} \, dw$$

$$= \tfrac{1}{2} \left\{ \int_{\gamma_n}^{\pi-(1/n)} \left(\sin \frac{w}{2}\right)^{\alpha-(1/2)} \left(\cos \frac{w}{2}\right)^{\beta+(1/2)} \phi(w). \, e^{iw(2n+\alpha+\beta+2)/2} \, dw \right.$$

$$- \int_{\gamma_n-\mu_n}^{\pi-(1/n)-\mu_n} \left(\sin \frac{w+\mu_n}{2}\right)^{\alpha-(1/2)} \left(\cos \frac{w+\mu_n}{2}\right)^{\beta+(1/2)}$$

$$\left. \times \, \phi(w+\mu_n) \, e^{iw(2n+\alpha+\beta+2)/2} \, dw \right\}.$$

Hence we have

$$|\text{ Real part of } E \,| \text{ or } |\text{ Imaginary part of } E\,|$$

$$\leqslant \tfrac{1}{2}(L_1 + L_2 + L_3 + L_4), \text{ say;}$$

where

$$L_1 = \int_{\gamma_n-\mu_n}^{\gamma_n} \left| \left(\sin \frac{w+\mu_n}{2}\right)^{\alpha-(1/2)} \left(\cos \frac{w+\mu_n}{2}\right)^{\beta+(1/2)} \phi(w+\mu_n) \right| dw$$

$$L_2 = \int_{\pi-\gamma_n-\mu_n}^{\pi-(1/n)} \left| \left(\sin\frac{w}{2}\right)^{\alpha-(1/2)} \left(\cos\frac{w}{2}\right)^{\beta+(1/2)} \phi(w) \right| dw$$

$$L_3 = \int_{\gamma_n}^{\pi-(1/n)-\mu_n} | \phi(w+\mu_n) - \phi(w) |$$

$$\times \left| \left(\sin\frac{w+\mu_n}{2}\right)^{\alpha-(1/2)} \left(\cos\frac{w+\mu_n}{2}\right)^{\beta+(1/2)} \right| dw$$

$$L_4 = \int_{\gamma_n}^{\pi-(1/n)-\mu_n} \left| \left(\sin\frac{w+\mu_n}{2}\right)^{\alpha-(1/2)} \left(\cos\frac{w+\mu_n}{2}\right)^{\beta+(1/2)} \right.$$

$$\left. - \left(\sin\frac{w}{2}\right)^{\alpha-(1/2)} \left(\cos\frac{w}{2}\right)^{\beta+(1/2)} \right| . | \phi(w) | \, dw.$$

But

$$L_1 = O(1) \int_{\gamma_n-\mu_n}^{\gamma_n} (w+\mu_n)^{\alpha-(1/2)+\delta} \, dw$$

$$= O(n^{-\alpha-(1/2)-\delta})$$

$$L_2 = O(1) \int_{\gamma_n-\mu_n}^{1/n} w^{\beta+(1/2)} \, dw$$

$$= O(n^{-\beta-(3/2)}) \quad \text{or} \quad = O(n^{-1}) \text{ by antipole condition}$$

$$L_3 = O(\mu_n^{\delta}) \int_{\gamma_n}^{\pi-(1/n)-\mu_n} (w+\mu_n)^{\alpha-(1/2)} \, dw$$

$$= O(n^{-\delta})$$

$$L_4 = O(\mu_n) \int_{\gamma_n}^{\pi-(1/n)-\mu_n} \left| \frac{d}{dw}\left\{\left(\sin\frac{w}{2}\right)^{\alpha-(1/2)} \left(\cos\frac{w}{2}\right)^{\beta+(1/2)}\right\} \right| w^{\delta} \, dw$$

$$= O(\mu_n) \int_{\gamma_n}^{a} w^{\alpha-(3/2)+\delta} \, dw + O(\mu_n) \int_{(1/n)+\mu_n}^{\pi-a} w^{\beta-(1/2)} \, | \phi(\pi - w) | \, dw.$$

(a any fixed number)

$$= \begin{cases} O(n^{-1}) & \text{if } \alpha - \tfrac{1}{2} + \delta > 0 \\ O(n^{-1}\log n), & \text{if } \alpha - \tfrac{1}{2} + \delta = 0 \\ O(n^{-\alpha-(1/2)-\delta}); & \text{if } \alpha - \tfrac{1}{2} + \delta < 0. \end{cases}$$

Therefore, we can approximate $J_{2\cdot1}$ as follows

$$J_{2\cdot1} = O(n^{\alpha+(1/2)}) \,.\, | \{L_1 + L_2 + L_3 + L_4\} |$$

or

$$\begin{aligned} J_{2\cdot1} &= O(n^{\alpha+(1/2)}) \{O(n^{-\alpha-(1/2)-\delta}) + O(n^{-\beta-(3/2)}) + O(n^{-\delta})\} \\ &\quad + \begin{cases} O(n^{\alpha-(1/2)}) \,.\, \log n \\ O(n^{-\delta}) \end{cases} \\ &= O(n^{\alpha+(1/2)-\delta}) + O(n^{\alpha-(1/2)} . \log n) \qquad \ldots(4.5) \end{aligned}$$

Combining (4.2) - (4.5), we get

$$\Sigma_n \frac{| S_n |}{n} < \infty.$$

Hence the series (2.2) is summable | C, 1 | . This completes the proof of the theorem.

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References

Bhatt, S. N. (1967). An aspect of local property of the absolute summability of the *r*th derived series. *Indian J. Math.*, **9** (1), 17-24.

Pandey, G. S. (1968). On Cesàro summability of Jacobi series. *Indian J. Math.*, **10** (2), 121-55.

Szegö, G. (1967). Orthogonal Polynomials. Colloquium Publication, New York.