ON TWO ANALOGUES OF COHEN'S THEOREM

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A duo-ring is one in which every left ideal is a two sided ideal and every right ideal is two sided. In the class of duo-rings, we characterize π-regular rings as the rings in which every prime ideal is maximal. The two analogues of Cohen's theorem for commutative rings are generalized to the class of left duo-rings.

INTRODUCTION

Following Feller (1958), Thierrin (1960) studied the properties of duo-rings. In this paper, we are mainly interested in extending the analogues of certain well-known theorems in commutative rings to the class of duo-rings. For instance we prove in Section 1 that a left duo-ring with 1 is left Noetherian if all its prime ideals are finitely generated. We also prove that in a left duo-ring with 1, if every prime ideal is principal then every ideal is principal. In Section 2, we prove a duo-ring R with 1 is π-regular if and only if every prime ideal in R is maximal. We also give examples to show that the condition 'every prime ideal is maximal' is neither a necessary nor a sufficient condition for a non-commutative ring to be π-ergular.

SECTION 1

Theorem 1—In a left duo-ring R with 1, if every prime ideal is a principal ideal, then every ideal is a principal ideal.

PROOF: Now from Thierrin (1960) (by a suitable argument for left duo-ring also), every prime ideal is completely prime. Suppose the theorem were not true, let F denote the non-empty collection of all ideals of R which are not principal. Partial order the elements in F by set inclusion. Then the union of an ascending chain of members of F is clearly a member of F. (This follows from the fact that any principal ideal in R is of the form Rx for some $x \in R$.) By Zorn's lemma, we get a maximal element M in F with reference to the property of not being a principal ideal. Clearly, M is not a prime ideal. Thus $\exists$ elements $a, b \in R$, $a, b \notin M$ such that $ab \in M$. Now, $M + Rb \supsetneq M$ and therefore

$$M + Rb = Rc \text{ for some } c \in R \quad \text{...(1)}$$

Let $K = \{r \in R / rb \in M\}$. K is a left ideal of R and therefore two sided.

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Also, $K \underset{\neq}{\supset} M$, since $a \notin M$ and $a \in K$. Therefore, $K = Rd$ for some $d \in R$, by the maximality of M in F. Claim $M = Rdc$.

Now for any $m \in M, m = xc$ for some $x \in R$ by (1). Let $b = s\bar{c}$ for some $s \in R$ by (1). Then $xb = x(sc) = (xs)c = (tx)c = t(xc) = tm \in M$ since $xs = tx$ (using R is left duo with 1), *i.e.*, $xb \in M$. So, $x \in K$. So $x = rd$ for some $r \in R$, *i.e.*, $m = rdc$. Thus, $M \subseteq Rdc$. Also, $c = m + rb$ for some $m \in M$ and $r \in R$ by (1). Therefore $dc = dm + drb = dm + r'(db)$ which $\in M$, $(dr = r'd$ for some r' in R since R is left duo) since $d \in K$, $db \in K$. Thus, $Rdc \subseteq M$, and hence $Rdc = M$ which is a contradiction to the fact that M is not principal. Hence the theorem.

Now we are going to prove the analogue of Cohen's theorem in the class of left duo-rings. The similar analogue for right duo-rings also holds.

Theorem 2—Let R be a left duo-ring with 1. Suppose in R every prime ideal is finitely generated, then every ideal in R is finitely generated.

PROOF: The general idea of the proof is the same as before. Suppose the theorem were not true, then in the non-empty set F of ideals which are not finitely generated, one can get a maximal element M in F with reference to the property of not being finitely generated. Clearly, M is not a prime ideal. Thus there exist elements, $a, b \in R$, such that $a, b \notin M$ and $ab \in M$. Now $M + Rb \underset{\neq}{\supset} M$, since $b \notin M$. Thus $M + Rb = Rx_1 + Rx_2 + \ldots + Rx_n$ for some finite set of generators $x_1, x_2, \ldots, x_n$. Now each $x_i = m_i + r_i b$ for some $m_i \in M$ and $r_i \in R$ by above equation (for $i = 1, 2, \ldots, n$). Then, clearly, $M + Rb = Rm_1 + Rm_2 + \ldots + Rm_n + Rb$. *i.e.*, $m_1, m_2, \ldots, m_n$, b is a finite set of generators for $M + Rb$. Let $K = \{r \in R / r\, b \in M\}$. K is a left ideal and thus two sided. Clearly $K \underset{\neq}{\supset} M$, since $a \in K$ and $a \notin M$. Thus, by the maximality of M in F, $K = Ry_1 + Ry_2 + \ldots + Ry_l$ for a finite set of generators $y_1, y_2, y_3 \ldots y_l$. Now, let m be any element in M. Then, $m = r_1 m_1 + r_2 m_2 + \ldots + r_n m_n + rb$ for some $r_i \in R$ and $r \in R$. Now, clearly $r \in K$. Thus, $r = r_1' y_1 + r_2' y_2 + \ldots + r_l' y_l$. Thus, $m = r_1 m_1 + r_2 m_2 + \ldots + r_n m_n + r_1'(y_1 b) + \ldots r_l'(y_l b)$, *i.e.*, $m_1, m_2, \ldots, m_n, y_1 b, y_2 b, \ldots, y_l b$ is a finite set of generators of M which is a contradiction.

SECTION 2

In this section, we characterise π-regular rings in the class of duo-rings with 1.

Definition—A ring R is said to be π-regular if for each a in R, $\exists\, x \in R$ and a positive integer n such that $a^n x a^n = a^n$.

Theorem 3—Let R be a duo-ring with 1. Then R is π-regular if and only if every prime ideal of R is a maximal ideal.

Proof: Suppose R is π-regular. Let P be a prime ideal in R. Let a be any element of R not in P. To show P is maximal, it suffices to show that the ideal generated by P and the element a contains the identity element 1 of R. Now, since R is π-regular $a^n x a^n = a^n$ for some positive integer n. [Since any prime ideal is completely prime in a duo-ring (Thierrin 1960) a is not nil-potent. For $a^m = 0 \in P$ implies $a \in P$ which is a contradiction.] Now, $a^n (xa^n - 1) = 0 \in P$. Since, $a^n \notin P$, $xa^n - 1 \in P$. *i.e.*, $1 = p + (xa^{n-1})\, a$ for some $p \in P$.

Sufficiency Part

Let a be any element in R. If $a^n = 0$ for some integer $n > 0$, then trivially $a^n \,.\, 1 \,.\, a^n = a^n$. Thus, we may assume a is not nil-potent.

Case 1—$aR = R$. In this case $ax = 1$ for some $x \in R$ and thus $a x a = a$.

Case 2—$aR \neq R$. Note that aR is a two sided ideal of R. In this case, using Zorn's lemma, one can show a belongs to a proper prime ideal P of R. Let $T =$ the set union of all prime ideals of R which contains a. Let $S = R - T$ (the set difference). Since, in a duo-ring every prime ideal is completely prime (Thierrin 1960), S is a multiplicatively closed set. Let $F =$ the multiplicatively closed system generated by the set $\{a\} \cup S$. Thus every elements in F is of the form $a^{n_1} s_1 a^{n_2} s_2 \ldots . a^{n_r} s_r$ where s_i $(i = 1, 2, \ldots, r) \in S$ and $n_i \geqslant 0$ $(i = 1, 2, \ldots, r)$. Notice that $\{a\} \cup S \subseteq F$.

Now, we assert that $0 \in F$. Suppose, this were not true partial order the collection of ideals disjoint with F by set inclusion. By Zorn's lemma, we get an ideal M which is maximal disjoint with F. One can show that this M is a prime ideal and hence maximal by hypothesis. Since $a \notin M$, the left ideal (since R is left duo, the left ideal is also two sided) generated by M and a is the whole ring R. Thus, $\exists$ elements $p \in M$, $r \in R$ such that $p + ra = 1$. From this, it follows that p is not in any prime ideal which contains a. Thus $p \in S \subseteq F$, which implies $p \in F \cap M = \phi$ which is a contradiction. Thus we have proved F contains 0, *i.e.*,

$$a^{n_1} s_1 a^{n_2} s_2 \ldots . a^{n_t} s_t = 0 \qquad \ldots(2)$$

where s_i $(i = 1, 2, \ldots, t) \in S$ and the integers $n_1, n_2, \ldots, n_t$ can all be assumed to be greater than zero. Since proper ideal cannot contain both a^{n_t} and s_t (then a prime ideal would contain both of them), the left ideal generated by a^{n_t} and s_t is the whole of R. Thus, $\exists$ elements $x, y \in R$ such that

$$xa^{n_t} + ys_t = 1. \qquad \ldots(3)$$

Now, multiplying equation (3) by

$a^{n_1} s_1 a^{n_2} s_2 \ldots . a^{n_t}$ from the left,

$$a^{n_1} s_1 a^{n_2} s_2 \ldots . a^{n_t} x a^{n_t} + a^{n_1} s_1 a^{n_2} \ldots . a^{n_t} y s_t = a^{n_1} s_1 a^{n_2} s_2 \ldots . a^{n_t}.$$

Now, using the property of duo, $ys_t = s_t y'$ for some $y' \in R$.

Thus

$$a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots a^{n_t} s_t = a^{n_1} s_1 a^{n_2} \ldots\ldots a^{n_t} s_t y' = 0 \text{ [using (2)].}$$

Therefore,

$$a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots a^{n_{t-1}} x\, a^{n_t} = a^{n_1} s_1 a^{n_2} \ldots\ldots s_{t-1} a^{n_t}$$

i.e.,

$$a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots a^{n_{t-1}} (s_{t-1} a^{n_t} x - s_{t-1})\, a^{n_t} = 0.$$

Now,

$$s_{t-1} a^{n_t} x - s_{t-1} \in S.$$

(This follows using the fact that s_{t-1} does not belong to any prime ideal that contains a.) Call

$$s_{t-1} a^{n_t} x - s_{t-1} = s'_{t-1}.$$

Thus,

$$a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots a^{n_{t-1}} s'_{t-1} a^{n_t} = 0. \qquad \ldots(4)$$

Now, note that a proper ideal cannot contain both the elements $a^{n_{t-1}}$ and s'_{t-1}. Thus, the left ideal generated by $a^{n_{t-1}}$ and s'_{t-1} is the whole of R. Thus, $\exists$ elements $x', y' \in R$ such that $x' a^{n_{t-1}} + y' s'_{t-1} = 1$.

Multiplying the above equation from the left and right by the elements

$a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots s_{t-2} a^{n_{t-1}}$ and a^{n_t} respectively, we get

$$a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots s_{t-2} a^{n_{t-1}} x' a^{n_{t-1}} a^{n_t} + a^{n_1} s_1 a^{n_2} \ldots\ldots s_{t-2} a^{n_{t-1}} y' s'_{t-1} a^{n_t}$$

$$= a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots\ldots s_{t-2} a^{n_{t-1}+n_t}$$

Now, using the duo-property,

$$y' (s'_{t-1} a^{n_t}) = (s'_{t-1} a^{n_t})\, z \text{ for some } z \in R,$$

Thus,

$$a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots s_{t-2} a^{n_{t-1}} y' s'_{t-1} a^{n_t}$$

$$= a^{n_1} s_1 a^{n_2} s_2 \ldots\ldots s_{t-2} a^{n_{t-1}} s'_{t-1} a^{n_t} = 0 \text{ using (4).}$$

Thus,

$$a^{n_1} s_1 a^{n_2} \ldots\ldots s_{t-2} a^{n_{t-1}} x' a^{n_{t-1}+n_t} = a^{n_1} s_1 a^{n_2} \ldots\ldots s_{t-2} a^{n_{t-1}+n_t}.$$

Once again, one can see that

$$s_{t-2} a^{n_{t-1}} x' - s_{t-2} \in S.$$

Call

$$s_{t-2}\, a^{n_{t-1}}\, x - s_{t-2} = s'_{t-2}.$$

Thus

$$a^{n_1} s_1\, a^{n_2} \ldots\ldots a^{n_{t-2}}\, s'_{t-2}\, a^{n_t+n_{t-1}} = 0. \qquad \ldots(5)$$

Now, proceeding in this manner, we finally get,

$$a^{n_1} s_1'\, a^k = 0 \text{ where } s_1' \in S, \text{ and } k \text{ is a suitable integer.}$$

By change of notation, $a^m s a^n = 0$ where $s \in S$.

Now, note that an ideal cannot contain both a^{m+n} and s. Thus, the left ideal generated (which is also two sided) by a^{m+n} and s contains 1.

i.e., $q\, a^{m+n} + rs = 1$ for some q, $r \in R$. Now multiplying the above equation from the left and right by a^m and a^n respectively,

$$a^m\, q\, a^{m+n+n} + a^m\, r\, sa^n = a^{m+n}.$$

Now, $r\,(sa^n) = (sa^n)\, r'$ for some $r' \in R$ (using right duo).

Thus, $a^m\, q\, a^{m+2n} = a^{m+n}$ (Since $a^m\, sa^n = 0$).

i.e., $a^m\, (qa^n)\, a^{m+n} = a^{m+n}$.

Now, using duo, $qa^n = a^n\, x$ for some $x \in R$.

Thus, $a^{m+n}\, x\, a^{m+n} = a^{m+n}$

i.e., R is π-regular.

Remark: Prof. Hyman Bass (personal communication) has pointed out that the property that every prime ideal is maximal, is in general neither a necessary, nor a sufficient condition for a ring to be π-regular.

For instance (Jacobson 1964, p. 211) there is an instance of a simple ring with identity which is a domain and not a division ring. Clearly, every prime ideal in this ring is maximal. However, this ring cannot be π-regular.

For the reverse direction, consider the ring R of all linear transformations on a countable infinite dimensional vector space V over a field F of characteristic 0. Then clearly R is a regular ring. Also the only non-zero ideal in R is the set of all linear transformations on V of finite rank. Here clearly (0) is a prime ideal of R and not maximal. Thus the condition is not necessary.

The author makes the following reasonable conjecture, *viz.*, "A ring R satisfying a polynomial identity is π-regular if and only if every prime ideal in R is maximal". It is not known whether the above stated conjecture is true or not.

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