FLOW OF A SECOND-ORDER FLUID IN A CURVED PIPE*

by H. G. SHARMA *and* ANAND PRAKASH, *Department of Mathematics, University of Roorkee, Roorkee*

(Communicated by S. K. Joshi, F.N.A.)

(*Received* 12 *March* 1975)

The flow of an incompressible second-order fluid in a slightly curved pipe has been discussed. The curvature of the pipe has been assumed to be small, that is, the radius of the circle in which the central line of the pipe is coiled is large in comparison with the radius of the cross-section. A solution is developed by successive approximations, the first approximation corresponding to the flow of Newtonian viscous liquid. The streamlines in the plane of symmetry and the projection of the streamlines on a normal section are compared with those for Newtonian liquid.

1. INTRODUCTION

Flow of non-Newtonian fluids through slightly curved pipes is of current interest in bio-mechanical and chemical industries such as flow of blood through arteries and flow through coiled pipes. The streamline motion of an incompressible Newtonian fluid in a slightly curved circular pipe has been solved theoretically by Dean (1927, 1928). The same problem has been extended by Jones (1960), Thomas and Walters (1963), Clegg and Power (1963) and Rathna (1967) for a non-Newtonian Reiner-Rivlin fluid, elastico-viscous fluid, Bingham fluid and a power-law fluid respectively. In the present paper, we have discussed the same problem to study the normal stress effects in the flow of a second-order fluid.

The constitutive equation for a second-order fluid as suggested by Coleman and Noll (1960) can be written as

$$T_{ij} = -P\delta_{ij} + 2\mu_1 d_{ij} + 2\mu_2 e_{ij} + 4\mu_3 d_i^{\alpha} d_{\alpha j}$$

where

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \text{ and } e_{ij} = \tfrac{1}{2}(a_{i,j} + a_{j,i} + 2v^{m}_{,i}\, v_{m,j}), \qquad \ldots(1.1)$$

T_{ij} is the stress-tensor, P the pressure; δ_{ij} the Kroneckar delta; a_i and v_i the acceleration and velocity vectors respectively; and μ_1, μ_2, μ_3 the material constants, known as Newtonian-viscosity, elastico-viscosity and cross-viscosity respectively.

*Presented at the 19th Congress on Theoretical and Applied Mechanics, India, 1974.

The equations of motion and continuity for steady motion in the absence of body forces are respectively given by

$$\rho v^j v_{i,j} = T^j_{i,j} \qquad ...(1.2)$$

and

$$v^i_{,i} = 0 \qquad ...(1.3)$$

where ρ is the density of fluid and comma denotes covariant differentiation.

2. FORMULATION

Consider the steady motion of an incompressible second-order fluid characterized by the constitutive eqn. (1.1) through a pipe of circular cross-section of radius a, with the line of curvature coiled in a circle of radius b, under a constant pressure-gradient. The coordinate system is the same as adopted by Dean (1927) and is illustrated in Fig. 1. OS is the axis of the anchor ring formed by the pipe wall, C is the centre of the cross-section of the pipe by a plane through OS, that makes an angle ϕ with a fixed plane and CO is perpendicular to OS of length b. The position of any point P in the section ϕ = constant can be specified by the orthogonal coordinates (R, θ, ϕ), where R is the distance CP and θ is the angle which CP makes with the line through C parallel to OS. The surface of the pipe is given by $R = a$, where a is the radius of the cross-section of the pipe.

The line element ds is thus given by

$$(ds)^2 = (dR)^2 + (R\,d\theta)^2 + (b + R\sin\theta)^2\,d\phi^2. \qquad ...(2.1)$$

We shall suppose that the motion of the liquid is due to a fall in the pressure along the pipe. Let $U(R, \theta)$, $V(R, \theta)$, $W(R, \theta)$ be the velocity components in the direction of R, θ and ϕ respectively and they are all independent of ϕ.

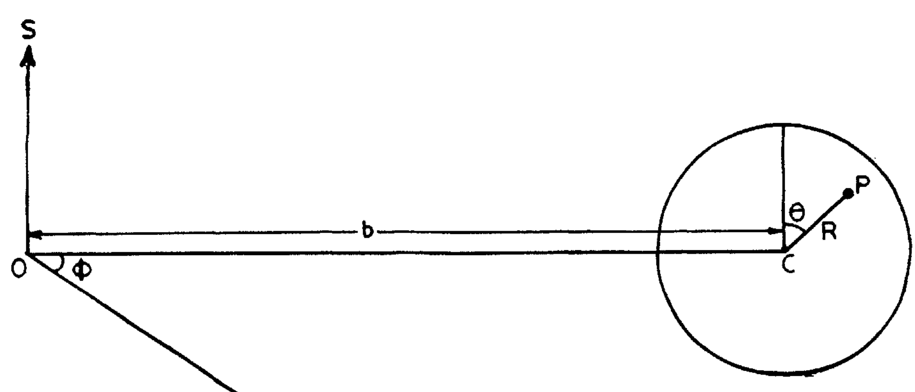

FIG. 1. The coordinate system (R, θ, ϕ) chosen to describe the motion in a curved pipe.

The equations of motion and continuity (1.2) and (1.3) take the form

$$\rho\left[U\frac{\partial U}{\partial R}-\frac{V^2}{R}-\frac{W^2\sin\theta}{b}\right]=\frac{\partial}{\partial R}T_{RR}+\frac{1}{R}\frac{\partial}{\partial\theta}T_{R\theta}+\frac{T_{RR}-T_{\theta\theta}}{R}-\frac{T_{\phi\phi}\sin\theta}{b}$$

$$\rho\left[U\frac{\partial V}{\partial R}+\frac{V}{R}\frac{\partial V}{\partial R}+\frac{UV}{R}-\frac{W^2\cos\theta}{b}\right]$$
$$=\frac{\partial}{\partial R}T_{R\theta}+\frac{1}{R}\frac{\partial}{\partial\theta}T_{\theta\theta}+\frac{2}{R}T_{R\theta}-\frac{T_{\phi\phi}\cos\theta}{b}$$

$$\rho\left[U\frac{\partial W}{\partial R}+\frac{V}{R}\frac{\partial W}{\partial\theta}\right]=\frac{\partial}{\partial R}T_{R\phi}+\frac{1}{R}\frac{\partial}{\partial\theta}T_{\theta\phi}+\frac{T_{R\phi}}{R}+\frac{2\sin\theta}{b}T_{R\phi}-\frac{1}{b}\frac{\partial P}{\partial\phi} \qquad \text{...(2.2)}$$

and

$$\frac{\partial U}{\partial R}+\frac{U}{R}+\frac{1}{R}\frac{\partial V}{\partial\theta}=0 \qquad \text{...(2.3)}$$

where we have assumed that the curvature of the pipe namely a/b is small and replace $\frac{1}{b+R\sin\theta}$ by $\frac{1}{b}$.

Primary motion — If the pipe were straight $\frac{a}{b}$ would vanish and the primary motion is specified by

$$U_0=0,\ V_0=0,\ W=W_0(R). \qquad \text{...(2.4)}$$

Substituting (2.4) in the equations of motion, we obtain the following velocity field and the pressure for the primary motion.

$$W_0=\frac{A}{4\mu_1}(a^2-R^2),\ P_0=\frac{3(2\mu_2+\mu_3)}{8\mu_1^2}A^2R^2-AZ+C \qquad \text{...(2.5)}$$

where A is a constant mean pressure gradient.

Secondary motion — Secondary flow is governed by the equations (2.2) and (2.3) and the boundary conditions are

$$R=a\,;\ U=V=W=0. \qquad \text{...(2.6)}$$

Following Dean (1927) the velocity components for the secondary motion induced by taking the curvature of the pipe, which is considered to be small, can be taken in the following form

$$U = -\frac{1}{R}\frac{\partial\psi}{\partial\theta},\ V = \frac{\partial\psi}{\partial R},\ W = W_0 + W_1(R, \theta),\ P = P_0 + P_1 \qquad \ldots(2.7)$$

where W_0 and P_0 are given by (2.5). Further, we shall take the secondary flow velocity components u, v, w to be of the order (a/b), so that the terms of the order $(a/b)^2$ and higher can be neglected in the equations of motion. Making use of (1.1) and (2.7), the equations of motion (2.2) reduce to

$$\begin{aligned}
&-\frac{A^2\rho}{16\mu_1^2}(a^2 - R^2)^2\frac{\sin\theta}{b} - \frac{A^2\mu_2}{4\mu_1^2}(3a^2 - 5R^2)\frac{\sin\theta}{b} \\
&\quad - \frac{A^2\mu_3}{8\mu_1^2}(3a^2 - 7R^2)\frac{\sin\theta}{b} \\
&\quad = -\frac{\partial P_1}{\partial R} - \frac{\mu_1}{R}\frac{\partial}{\partial\theta}\left(\frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R}\right) \\
&\quad - \frac{\mu_2}{\mu_1}A\left(4\frac{\partial W_1}{\partial R} + 2R\frac{\partial^2 W_1}{\partial R^2} + \frac{1}{R}\frac{\partial^2 W_1}{\partial\theta^2}\right) \\
&\quad - \frac{\mu_3}{2\mu_1}A\left(4\frac{\partial W_1}{\partial R} + 2R\frac{\partial^2 W_1}{\partial R^2} + \frac{1}{R}\frac{\partial^2 W_1}{\partial\theta^2}\right) \\
&-\frac{A^2\rho}{16\mu_1^2}(a^2 - R^2)\frac{\cos\theta}{b} - \frac{A^2\mu_2}{4\mu_1^2}(3a^2 - 5R^2)\frac{\cos\theta}{b} \\
&\quad - \frac{A^2\mu_3}{8\mu_1^2}(3a^2 - 7R^2)\frac{\cos\theta}{b} \\
&\quad = -\frac{1}{R}\frac{\partial P_1}{\partial\theta} + \mu_1\frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R}\right) \\
&\quad - \frac{\mu_2}{\mu_1}A\frac{\partial}{\partial\theta}\left(\frac{\partial W_1}{\partial R} + \frac{2W_1}{r}\right) - \frac{\mu_3}{2\mu_1}A\frac{\partial}{\partial\theta}\left(\frac{\partial W_1}{\partial R} + \frac{2W_1}{R}\right) \\
&\frac{\rho A}{2\mu_1}\frac{\partial\psi}{\partial\theta} + \frac{3}{2}\frac{AR}{b}\sin\theta = -\frac{\partial P_1}{\partial z} + \mu_1\left(\frac{\partial^2 W_1}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 W_1}{\partial\theta^2} + \frac{1}{R}\frac{\partial W_1}{\partial R}\right) \\
&\quad + \frac{\mu_2}{2\mu_1}A\frac{\partial}{\partial\theta}\left(\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{R}\frac{\partial\psi}{\partial R}\right) \\
&\quad + \frac{\mu_3}{2\mu_1}A\frac{\partial}{\partial\theta}\left(\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2}\right)
\end{aligned} \qquad \ldots(2.8)$$

where we have written, in conformity with (2.7), $z = b\phi$ or $\frac{1}{b}\frac{\partial}{\partial\phi} = \frac{\partial}{\partial z}$.

3. Solution

The equations (2.8) suggest the solution in the form

$$\psi = \frac{Aa^4}{4\mu_1 b}\psi(r)\cos\theta,\quad W_1 = \frac{Aa^3}{4\mu_1 b}w(r)\sin\theta,\quad P_1 = \frac{Aa^2}{4b}p(r)\sin\theta \qquad \ldots(3.1)$$

where ψ, w and p are the dimensionless parameters and functions of $r\left(=\frac{R}{a}\right)$ only. Substituting (3.1) in eqns. (2.8), following equations are obtained:

$$\left.\begin{aligned}
&-R_e(1-r^2)^2-4T(3-5r^2)-2K(3-7r^2)=-\frac{\partial p}{\partial r}+\frac{\Delta\psi}{r}\\
&\qquad\qquad -4T\left(4\frac{\partial w}{\partial r}+2r\frac{\partial^2 w}{\partial r^2}-\frac{w}{r}\right)\\
&\qquad\qquad -2K\left(4\frac{\partial w}{\partial r}+2r\frac{\partial^2 w}{\partial r^2}-\frac{w}{r}\right)\\
&-R_e(1-r^2)^2-4T(3-5r^2)-2K(3-7r^2)\\
&=-\frac{p}{r}+\frac{d}{dr}(\Delta\psi)-4T\left(\frac{\partial w}{\partial r}+\frac{2w}{r}\right)-2K\left(\frac{\partial w}{\partial r}+\frac{2w}{r}\right)\\
&-2R_e\psi+6r=\Delta w-2T\,\Delta\psi-2K\,\Delta\psi
\end{aligned}\right\}\quad\ldots(3.2)$$

where R_e (Reynold's number), T and K are dimensionless parameters given by

$$R_e=\frac{\rho a^3 A}{4\mu_1^2},\ T=\frac{Aa\mu_2}{4\mu_1^2},\ K=\frac{Aa\mu_3}{4\mu_1^2}\text{ and }\Delta\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}.\quad\ldots(3.3)$$

The boundary conditions are

$$\left.\begin{aligned}
&w=\psi=\frac{d\psi}{dr}=0\text{ at }r=1,\\
&w,\ \frac{\psi}{r},\ \frac{d\psi}{dr}\text{ must be finite for }0\leqslant r\leqslant 1,\\
&\text{and}\\
&\frac{\psi}{r}=\frac{d\psi}{dr}=0\text{ at }r=0.
\end{aligned}\right\}\quad\ldots(3.4)$$

Eliminating p from equations (3.2), we obtain

$$4R_e(1-r^2)\,r+40Tr+28Kr=\Delta\,(\Delta\psi)+(4T+2K)\,\Delta w.\quad\ldots(3.5)$$

Eliminating w between (3.5) and (3.2), we get

$$\begin{aligned}4R_e(1-r^2)\,r+16Tr+16Kr&=\Delta^2\psi-4R_e\psi(2T+K)\\&\quad+4\Delta\psi(2T^2+K^2)+12TK\,\Delta\psi.\end{aligned}\quad\ldots(3.6)$$

When $S=(T+K)$ is sufficiently small, the solutions of the equations (3.6) and (3.2) can be attempted by expanding ψ, p and w in ascending powers of S as

$$\left.\begin{aligned} \psi &= \psi_0 + S\psi_1 + S^2\psi_2 + \ldots, \\ p &= p_0 + Sp_1 + S^2p_2 + \ldots. \\ \text{and} \qquad & \\ w &= w_0 + Sw_1 + S^2w_2 + \ldots . \end{aligned}\right\} \qquad \ldots(3.7)$$

Substituting (3.7) in (3.6) and (3.2) and equating the coefficients of S^0, S^1 etc., we have

$$\left.\begin{aligned} \triangle^2\psi_0 &= 4R_e(1 - r^2)\, r \\ \triangle^2\psi_1 &= 6R_e\psi_0 + 16r \end{aligned}\right\} \qquad \ldots(3.8)$$

and

$$\left.\begin{aligned} \triangle w_0 &= 6r - 2R_e\psi_0 \\ \triangle w_1 &= 2\,\triangle\psi_0 - 2R_e\psi_1. \end{aligned}\right\} \qquad \ldots(3.9)$$

Each successive approximation to ψ and w must satisfy the boundary conditions (3.4).

The solution of (3.8) and (3.9) has been obtained after integration as

$$\psi_0 = \frac{R_e}{288}(4r + 6r^5 - 9r^3 - r^7) \qquad \ldots(3.10)$$

$$\psi_1 = \frac{R_e^2}{460800}(91r - 230r^3 + 200r^5 - 75r^7 + 15r^9 - r^{11}) + \left(\frac{r^5}{12} - \frac{r^3}{6} + \frac{r}{12}\right) \qquad \ldots(3.11)$$

and

$$w_0 = \frac{R_e^2}{11520}(19r - 40r^3 + 30r^5 - 10r^7 + r^9) + \left(\frac{3}{4}r^3 - \frac{3}{4}r\right) \qquad \ldots(3.12)$$

$$w_1 = \frac{R_e}{288}(11r - 24r^3 + 16r^5 - 3r^7) - \frac{R_e^3}{230400}\left(-\frac{1727}{336}r + \frac{91}{8}r^3 - \frac{115}{12}r^5 + \frac{25}{6}r^7 - \frac{15}{16}r^9 + \frac{1}{8}r^{11} - \frac{1}{168}r^{13}\right). \qquad \ldots(3.13)$$

Similarly, we get

$$p_0 = \frac{R_e}{12}(9r + 2r^5 - 6r^3) \qquad \ldots(3.14)$$

and

$$p_1 = \left(\frac{173}{12}r - \frac{89}{4}r^3\right) + \frac{R_e^2}{460800}(-8680r + 38400r^3 - 43200r^5 + 19200r^7 - 2400r^9). \qquad \ldots(3.15)$$

4. Discussion of the Results

(i) *Stream-lines* — The stream-lines in the central plane are given by

$$\frac{dR}{U} = \left(\frac{R_e + R\sin\theta}{w}\right) d\phi, \quad \left(\theta = \pm\frac{\pi}{2}\right). \qquad \text{...(4.1)}$$

To a sufficient approximation, (4.1) can be written as

$$\pm\frac{d\phi}{dR} = \frac{72}{R_e(1-r^2)\left(1-\frac{r^2}{4}\right)} - \frac{72S}{R_e(1-r^2)\left(1-\frac{r^2}{4}\right)} \times\left[(91-48r^2+13r^4-r^6)\frac{R_e}{6400}+\frac{6}{R_e}\right] \qquad \text{...(4.2)}$$

It follows from (4.2), after integration, that

$$\pm\phi = \phi_0 + S\phi_1 \qquad \text{...(4.3)}$$

$$\phi_0 = \frac{24}{R_e}\log\left\{\frac{(1+r)^2\left(1-\frac{r}{2}\right)}{(1-r)^2\left(1+\frac{r}{2}\right)}\right\} \qquad \text{...(4.4)}$$

and

$$\phi_1 = -\frac{9}{50}r + \left(\frac{72}{R_e^2}+\frac{129}{1600}\right)\frac{r}{\left(1-\frac{r^2}{4}\right)} - \left(\frac{384}{R_e^2}+\frac{11}{20}\right)\log\left(\frac{1+r}{1-r}\right) + \left(\frac{264}{R_e^2}+\frac{281}{1600}\right)\log\frac{\left(1+\frac{r}{2}\right)}{\left(1-\frac{r}{2}\right)}. \qquad \text{...(4.5)}$$

For numerical illustration, we take

$$R_e = 63.3, \frac{a}{b} = \frac{1}{3} \qquad \text{...(4.6)}$$

as considered by Dean (1927). The values of ϕ_0 and ϕ_1 in degrees for corresponding values of r are given in Table I.

Table I

r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ϕ_0	0	6.6	13.3	20.3	28.0	36.6	46.8	59.5	77.0	106.8
ϕ_1	0	−6.45	−13.18	−20.17	−27.56	−35.80	−45.32	−56.78	−72.54	−98.5

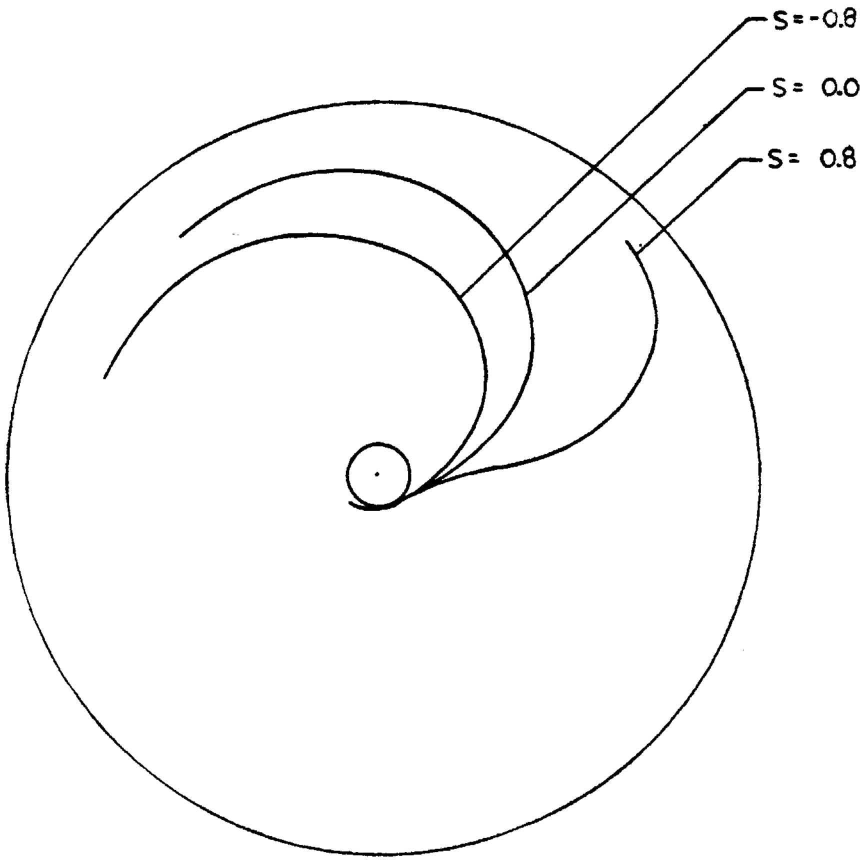

FIG. 2. Stream-lines in the central plane when $R_e = 63.3$.

The contribution of ϕ_1 to ϕ is positive when $S < 0$ and negative when $S > 0$. It is assumed that (3.7) is convergent when $| S | < 1.5$ as seen from the first few terms. The form of the stream-lines for the values $S = \pm 0.8$ are shown in Fig. 2. It is seen that positive S is to decrease the angular distance and the fluid particles travel from inner edge to near the outer edge and negative S is to increase the angular distance.

(ii) *Sream-line projections* —It will be of interest to draw also the curves of intersection of the surface $\psi =$ constant with a normal section $\phi =$ constant, the curves have the polar equation

$$\sec\theta = kr(1-r^2)^2\left(1-\frac{r^2}{4}\right)\left[1+S\left(1-\frac{r^2}{4}\right)^{-1} \times\left\{(91-48r^2+13r^4-r^6)\frac{R_e}{6400}+\frac{6}{R_e}\right\}\right] \qquad \ldots(4.7)$$

where k is an arbitrary constant. When $S = 0$ and $T = 0$, the solution is in agreement with Dean (1927) and Jones (1960) respectively. For $R_e = 0$, we get the case of creeping flow [Das and Roy (1974)]. This relation between r and θ is independent of R_e and (a/b).

From equations (2.7), (3.1) and (3.7), we see that to a sufficient approximation V vanishes, for all values of θ, when

$$4 - 23r^2 + 7r^4 + S\left\{(91 - 599r^2 + 401r^4 - 124r^6 + 11r^8)\frac{R_e}{1600} + (1 - 5r^2)\frac{24}{R_e}\right\} = 0. \quad ...(4.8)$$

When $S = 0$, the relevant solution is $r = 0.429$ independent of R_e. Taking $R_e = 63.3$, the relevant solutions of (4.8), in particular cases $S = -1.12$ and $S = 1.12$ are $r = 0.58$ and $r = 0.421$ respectively. Thus there is a particular stream-line in the form of a circle $r =$ constant, $\theta = n\pi$ since both U and V vanish at the points $r = 0.58$, $\theta = n\pi$ when $S = -1.12$; $r = 0.43$, $\theta = n\pi$ when $S = 0$ and $r = 0.421$, $\theta = n\pi$ when $S = 1.12$. Therefore, for any particular values of S, there is a limiting surface $\psi = \psi_l$ (ψ_l dependent on S) which takes the degenerate form of a single circular stream-line in a plane parallel to the central plane. The intersection of $\psi = \psi_l$ with a section $\phi =$ constant are denoted in Figs. 3 and 4. The line $\psi = \psi_l$ is defined by $k = 50$ when $S = -1.12$, by $k = 3.7$ when $S = 0$ and by $k = 1.9$ when $S = 1.12$; $k = \infty$ for all values of S, corresponds to the pipe wall. Denoting by k_l the values of k corresponding to $\psi = \psi_l$, the surface corresponding $k/k_l = 1.8$, 3.5 and ∞ (along the pipe wall) are shown in Figs. 3 and 4. It is clear that as S increases from -1.12 to 1.12 the distance from the central plane of the limiting stream line $\psi = \psi_l$ decreases. Therefore we conclude that as S decreases the surface $\psi =$ constant becomes closer to the limiting circular stream-lines.

(iii) *Rate of outflow* — The rate of outflow through the pipe is

$$\int_{r=0}^{1}\int_{\theta=0}^{2\pi} a^2 r W \, dr \, d\theta \quad ...(4.9)$$

To our order of approximations, w_1 makes no contribution to this integral and the rate of out flow is the same as if the pipe were straight. Thus to determine the effects of elastico-viscosity and cross-viscosity on the rate of outflow, we have to consider the terms $O(a/b)^2$ as discussed by Dean (1928) and Thomas and Walters (1963). To this approximation, the flux per second is given by

$$F_c = 2\pi a^2 \int_0^1 r^2 \, W dr.$$

$$= \frac{\pi a^2 W_0}{2}\left[1 - \left(\frac{S}{576}\right)^2 (0.04614)\right] \quad ...(4.10)$$

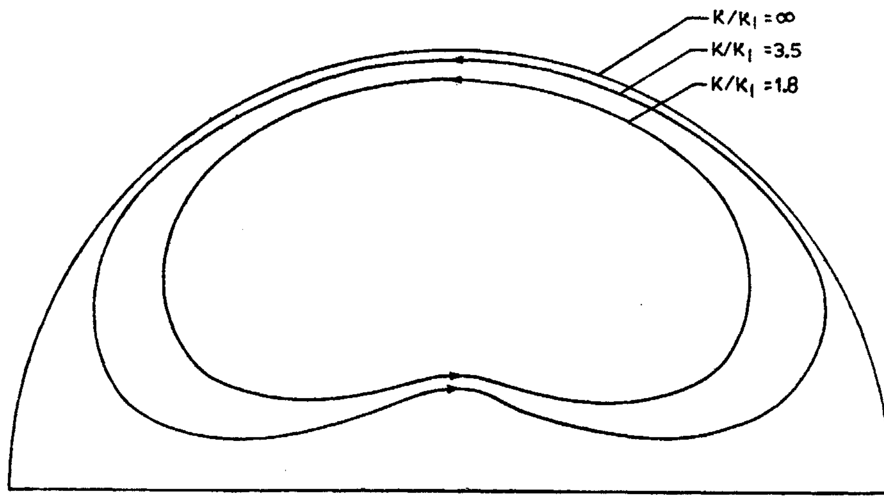

FIG. 3. Curves of intersection of the surfaces ψ = const. with a normal section ϕ = constant when $S = -1.12$ and $R_e = 63.3$.

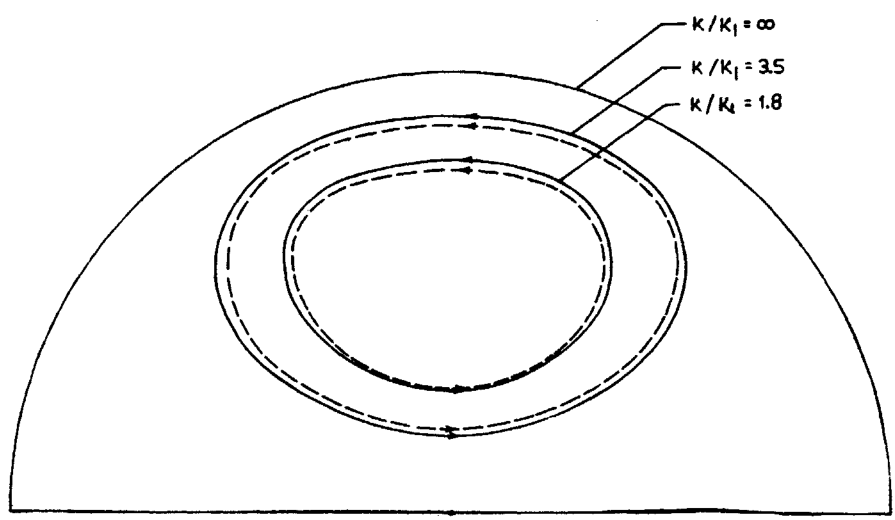

FIG. 4. Curves of intersection of the surface ψ = const. with a normal section ϕ = const. when $S = 1.12$ and $R_e = 63.3$ ---- represents for the Newtonian fluid ($S = 0$).

If the pipe were straight the flux per second (F_s) would be $\dfrac{\pi W_0 a^2}{2}$.

$$F_c/F_s = 1 - \left(\frac{S}{576}\right)^2 (0.04614)$$

represents the ratio of the rates of flow in two pipes of the same cross-section which are respectively curved and straight. It is evident that the elastico-viscosity and cross-viscosity in the fluid decrease the dependence of the flux through the pipe on the curvature of the pipe, to the second-order in the curvature.

(iv) To discuss the problem further, we can obtain the stresses in the flow of the second-order fluids. The stresses acting across the bounding surface ($r = 1$) are $-(T_{rr})_{r=1}$ and $(T_{rz})_{r=1}$. Neglecting the terms $O\left(\frac{a}{b}\right)^2$, we get

$$T_{rr} - T^0_{rr} = \left[- p + 2\left(\frac{1}{r}\frac{d\psi}{dr} - \frac{\psi}{r^2}\right) - 8T\left\{r\frac{dw}{dr} - r(1 - r^2)\right\} - 4K\left\{r\frac{dw}{dr} - r(1 - r^2)\right\}\right] \frac{Aa^2 \sin\theta}{4b} \quad \ldots(4.11)$$

and

$$T_{rz} - T^0_{rz} = \left[\frac{dw}{dr} - 1 + r^2 - 2T\left(\frac{d\psi}{dr} - \frac{\psi}{r}\right) - 4K\left(\frac{d\psi}{dr} - \frac{\psi}{r}\right)\right] \frac{Aa^2}{4b} \sin\theta \quad \ldots(4.12)$$

where T^0_{rr} and T^0_{rz} are the stresses when the pipe were straight and the fluid flows under the same axial pressure gradient.

(v) If δD denotes the axial drag on a length δz of the curved pipe, then

$$\delta D = \int_{\theta=0}^{2\pi} T_{rz}\, a\, d\theta\, \delta z = 2\pi a\, T^0_{rz}\, \delta z \quad \ldots(4.13)$$

which is the same as for a straight pipe except that the direction of δD varies with the axial distance z.

From (4.11) we obtain the normal stress effects given by

$$- [T_{rr} - T^0_{rr}] = \left[\frac{5R_e}{12} + K(2.3 - 8.1\alpha)\right] \frac{Aa^2}{4b} \sin\theta \text{ where } T = \alpha K. \quad \ldots(4.14)$$

Markovitz and Coleman (1964) have shown that 5.4% solution of polyisobutylene in cetane at 30°C behaves as a second-order fluid. The values of the material constants have been found to be $\mu_1 = 18.5$, $\mu_2 = -.2$, $\mu_3 = 1.0$ (in C.G.S. units). For the purpose of numerical calculations, the Reynolds number R_e may be taken fixed at 63.3 and $\alpha = -0.2$. The equation (4.14) therefore reduces to

$$- [T_{rr} - T^0_{rr}]_{r=1} = (26.38 + 3.92K) \frac{Aa^2}{4b} \sin\theta \quad \ldots(4.15)$$

which shows that the effect of a negative K is to set up the normal pressure which tends to keep the boundary section circular and that a positive K tends to make the pipe wall collapse.

ACKNOWLEDGEMENT

The authors are grateful to Professor C. Prasad for encouragement. One of the authors (AP) is grateful to U.P. State C.S.I.R. for a fellowship under a research scheme.

REFERENCES

Clegg, D. B., and Power, G. (1963). Flow of a Bingham fluid in a slightly curved pipe. *Appl. scient. Res.*, **12**, 199.

Coleman, B. D., and Noll, W. (1960). An approximation theorem for functionals with applications in continuum mechanics. *Arch. ration. Mech. Analysis*, **6**, 355.

Das, S. C., and Roy, N. (1974). Creeping flow of an incompressible second-order fluid through a curved pipe of small curvature. *Proc.* 19th *Cong. Theor. and appl. Mech., India.*

Dean, W. R. (1927). Note on the motion of fluid in a curved pipe. *Phil. Mag.*, **20**, 208.

———— (1928). The stream-line motion of fluid in a curved pipe. *Phil. Mag.*, **5**, 673.

Jones, J. R. (1960). Flow of a Non-Newtonian fluid in a curved pipe. *Q. Jl Mech. appl. Math.*, **13**, 428.

Markovitz, H., and Coleman, B. D. (1964). Incompressible second-order fluids. *Adv. appl. Mech.*, **8**, 69.

Rathna, S. L. (1967). Flow of a power law fluid in a curved pipe of circular cross-section. *Proc. Sem. Fluid Mech., India*, 378.

Thomas, R. H., and Walters, K. (1963). On the flow of an elastico-viscous liquid in a curved pipe under a pressure gradient. *J. Fluid Mech.*, **16**, 228.