SOME RESULTS ON p-VALUED ENTIRE ALGEBROID FUNCTIONS

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Some results of Valiron, Pólya and Shah concerning the growth of the maximum modulus relative to the counting function of an entire function have been extended to p-valued entire algebroid function $y(z)$: Following Shah, e.v.s., i.e. exceptional value in sense of Shah for $y(z)$ has been defined. Finally, it has been deduced as a consequence of one of our results that if $y(z)$ is a p-valued entire algebroid function of nonintegral order, then there are at most $p-1$ e.v.s.

1. INTRODUCTION

Valiron (1914, 1923, 1924) and Pólya (1923) have proved that if $f(z)$ is an entire function of finite non-integral order, then

$$\lim_{r\to\infty} \sup \frac{n(r)}{\log M(r)} > 0. \qquad \ldots(1.1)$$

This result is no longer true for integral order. But Shah (1940, 1941, 1942) has proved, for a function of integral order whose genus is the same as that of its canonical product, that

$$\lim_{r\to\infty} \sup \frac{n(r)\,\varphi(r)}{\log M(r)} = \infty, \qquad \ldots(1.2)$$

for any positive continuous increasing function $\phi(r)$, such that

$$\int_1^\infty \frac{dt}{t\,\varphi(t)} \qquad \ldots(1.3)$$

converges. However, Boas (1953) gave a shorter proof of the results (1.1) and (1.2) in a slightly more general form assuming that $\varphi(r)$ is only positive and integral (1.3) exists.

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Our purpose in this paper is to obtain the above results, along with a few more, for a p-valued entire algebroid function.

2. Notations and Terminology

Let $y(z)$ be a p-valued entire algebroid function in $|z| < \infty$, defined by an irreducible equation

$$F(z, y) = y^n + A_1(z)\, y^{n-1} + \ldots + A_n(z) = 0,$$

where $A_1, A_2, \ldots, A_n$ are entire functions without any common zeros.

Let us write

$$M(r, y) = \max_{|z|=r} \max_{1 \leqslant j \leqslant p} |y_j|,$$

where y_j denotes the jth determination of y. Then, the order ρ of $y(z)$ is defined as

$$\limsup_{r\to\infty} \frac{\log\log M(r, y)}{\log r} = \rho \quad (0 \leqslant \rho \leqslant \infty).$$

If $0 < \rho < \infty$, then there exists a proximate order* $\rho(r)$ of the function $y(z)$ satisfying the conditions :

(i) $\lim_{r\to\infty} \rho(r) = \rho$

(ii) $\lim_{r\to\infty} r\rho'(r) \log r = 0$

(iii) $\limsup_{r\to\infty} \dfrac{\log M(r, y)}{r^{\rho(r)}} = 1.$

By $n(r, a, y)$ we mean the quantity $n(r, a, F(z, a))$, the number of the roots of the equation $F(z, a) = 0$ in the disc $|z| \leqslant r$. We shall also be using the following†:

$$N(r, a, y) = \int_0^r \frac{n(t, a, y)}{t}\, dt,$$

$$N_q(r, a, y) = \int_0^r \frac{n(t, a, y)}{t^{q+1}}\, dt,$$

where q is a positive integer.

*For its existence and various properties see Cartwright (1962), Levin (1964), and Shah (1946).

†Throughout, we consider those points a for which $F(0, a) = 1$, and so the integrals converge at the lower limits.

3. Statements and Discussions of the Results

Theorem 1 — Let $y(z)$ be p-valued entire algebroid function of nonintegral order ρ. Then, there is at least one a_j among p-different fininte numbers $a_1, a_2, \ldots, a_p$ satisfying

$$\lim_{r\to\infty} \sup \frac{pn(r, a_j, y)}{\log M(r, y)} \geqslant \frac{\sin \pi(\rho - q)}{3e\pi(q+1)(2+\log q)}.$$

where $q = [\rho]$.

With the same hypothesis as of theorem 1, we also have :

Theorem 2 — There is at least one a_j among p-different finite numbers $a_1, a_2, \ldots, a_p$ satisfying

$$\lim_{r\to\infty} \sup \frac{pN(r, a_j, y)}{\log M(r, y)} \geqslant \frac{\sin \pi(\rho - q)}{3e\pi(q+1)^2(2+\log q)}.$$

Theorem 3 — There is at least one a_j among p-different finite numbers $a_1, a_2, \ldots, a_p$ satisfying

$$\lim_{r\to\infty} \sup \frac{p\,N_q(r, a_j, y)}{\log M(r, y)} \geqslant \frac{\sin \pi(\rho - q)}{3e\pi(q+1)(2+\log q)(\rho - q)}.$$

Remark : The result in Theorem 2 extends an earlier result of Ozawa (1970), obtained for a p-valued entire algebroid function of order ρ, $0 < \rho < 1$.

Corollary — Under the hypothesis of theorem 1, there is at least one a_j among p-different finite numbers $a_1, a_2, \ldots, a_p$ such that

$$\lim_{r\to\infty} \sup \frac{pn(r, a_j, y)\varphi(r)}{\log M(r, y)} = \infty \qquad \ldots(3.1)$$

where $\varphi(r)$ is any increasing function of r. The same conclusion if $n(r, a_j, y)$ is replaced by $N(r, a_j, y)$ or $r^q N_q(r, a_j, y)$.

Remark : The result (3.1) is analogous to (1.1).

To state the next result more precisely, let us consider

$$F(z, a_\nu) = e^{Q_\nu(z)} P_\nu(z),$$

where $P_\nu(z)$ is the cononical product formed by the zeros of $F(z, a_\nu)$ and $Q_\nu(z)$ is a polynomial. Let q_ν be the genus of $F(z, a_\nu)$ and a_ν the degree of $Q_\nu(z)$. Let s_ν be the genus of $P_\nu(z)$. Put $q = \max_{1\leqslant\nu\leqslant p} q_\nu$, $d = \max_{1\leqslant\nu\leqslant p} d_\nu$ and $s = \max_{1\leqslant\nu\leqslant p} s_\nu$. By the definition of genus $q_\nu = \max_{1\leqslant\nu\leqslant p} (d_\nu, s_\nu)$. Thus $q = \max(d, \rho)$. In case $d \leqslant \rho$, then $q = \rho$.

As an analogue of (1.2) we should have the following for a p-valued entire algebroid function.

Theorem 4 — Let $y(z)$ be of integral order ρ, and $a_1, a_2, \ldots, a_p$ be p different numbers for which $q = s$. Then, there exists at least one a_j among $a_1, a_2, \ldots, a_p$ such that

$$\lim_{r \to \infty} \sup \frac{pn(r, a_j, y)}{\log M(r, y)} = \infty$$

where φ is a positive function satisfying (1.3).

However, we are unable to prove this theorem and it is still an open problem.

Following Shah (1951), we may define a value α $(0 \leqslant |\alpha| < 1)$ as an exceptional value in sense of Shah for a p-valued entire algebroid function $y(z)$ if

$$\lim_{r \to \infty} \inf \frac{\log M(r)}{n(r, \alpha, y)\, \zeta(r)} > 0$$

for some $\zeta \in S$, where S denotes the set of all positive non-decreasing functions $\zeta(x)$ for which the integral (1.3) exists. Briefly, α is e.v.s.

As a consequence of Theorem 1, we have

Proposition — If $y(z)$ is a p-valued entire algebroid function of nonintegral order, then there are atmost $p - 1$ e.v.s.

Remark : If there are p-different e.v.s. of a p-valued entire algebroid function, then the function is necessarily of integral order.

4. Proofs of Theorems 1, 2, and 3

We have

$$\begin{aligned} \log M(r, y) &= \max_{|z|=r} \max_{1 \leqslant \nu \leqslant p} \log |y_\nu(z)| \\ &\leqslant \max_{|z|=r} \max_{1 \leqslant \nu \leqslant p} \log |y_\nu(z)| \\ &\leqslant \max_{|z|=r} \sum_{\nu=1}^{p} \log^+ |y_\nu(z)| . \end{aligned}$$

Let us write

$$A(z) = \max \{1, |A_1|, \ldots, |A_n|\}$$

$$g(z) = \max \{|g_1|, \ldots, |g_n|\}$$

with $g_\nu(z) = F(z, a_\nu)$. Then, by Valiron's (1914) arguments, we find that

$$\sum_{\nu=1}^{p} \log^+ | y_\nu(z) | \leqslant \log | A(z) | + O(1)$$
$$\leqslant \log | g(z) | + O(1).$$

Therefore

$$\log M(r, y) \leqslant \log \{ \max_{|z|=r} g(z)\}$$
$$= \log \{ \max_{1\leqslant\nu\leqslant p} \max_{|z|=r} g_\nu(z)\}$$
$$= \max_{1\leqslant\nu\leqslant p} \log M(r, g_\nu).$$

It is well known that (see Levin 1964, p. 57)

$$\log M(r, g_\nu) \leqslant K(g) \int_0^\infty \frac{n(t, 0, g_\nu)}{r^{q+1}(t+r)} r^{q+1}\, dt + O(r^q) \qquad \ldots(4.1)$$

where $K(q) = 3e(q+1)(2+\log q)$ since $q_\nu \leqslant q$. Hence

$$\log M(r, y) \leqslant K(q) \max_{1\leqslant\nu<p} \int_0^\infty \frac{n(t, 0, g_\nu)}{t^{q+1}(t+r)}\, dt + O(r^q)$$
$$= K(q) \max_{1\leqslant\nu\leqslant p} \int_0^\infty \frac{n(t, a_\nu, y)}{t^{q+1}(t+r)}\, dt + O(r^q)$$
$$\leqslant K(q) \max_{1\leqslant\nu\leqslant p} p \int_0^\infty \frac{n(t, a_\nu, y)}{t^{q+1}(t+r)}\, dt + O(r^q). \qquad \ldots(4.2)$$

Assume, for all ν that

$$\limsup_{r\to\infty} \frac{pn(r, a_\nu, y)}{\log M(r, y)} < \frac{\sin \pi(p, q)}{3e\,\pi(q+1)(2+\log q)}.$$

Then

$$\frac{pn(r, a_\nu, y)}{\log M(\mathrm{r}, y)} < \frac{\sin \pi(\rho - q)}{3e\,\pi(q+1)(2+\log q)} - \epsilon = U$$

for all $r \geqslant r_0$. Thus

$$\log M(r, y) < r^{q+1} K(q)\, U \int_0^\infty \frac{\log M(r, y)}{t^{q+1}(t+r)}\, dt$$

$$< r^{q+1} K(q)\, U \int_0^\infty \frac{t^{\rho(t)}}{t^{q+1}(t+r)}\, dt.$$

Putting $t = ur$ in the above, we have

$$\log M(r, y) < r^{\rho(r)} K(q)\, U \int_0^\infty \frac{U^{\rho-q-1}}{u+1}\, du + O(r^q)$$

$$= \frac{r^{\rho(r)} K(q)\, U\pi}{\sin \pi(\rho - q)} + O(r^q),$$

since $\rho - q - 1$. Hence

$$1 \leqslant \frac{3e(2 + \log q)(q+1)\pi}{\sin \pi(\rho - q)} U < 1,$$

which is a contradiction. This proves Theorem 1.

The proof of Theorems 2 and 3 follow on the same lines as that of the proof of Theorem 1 by using the following estimates in place of (4.1):

$$\log M(r, g_\nu) \leqslant (q+1) K(q)\, r^{q+1} \int_0^\infty \frac{N(t, 0, g_\nu)}{t^{q+1}(t+r)}\, dt + O(\nu)$$

and

$$\log M(r, g_\nu) \leqslant K(q)\, r^{q+1} \int_0^\infty \frac{N_g(t, 0, g_\nu)}{(t+r)^2}\, dt + O(r^q).$$

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