DETERMINATION OF PHASE SHIFTS FOR YUKAWA POTENTIAL FUNCTION

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(Communicated by P. L. Bhatnagar, F.N.A.)

(*Received* 13 *March* 1975; *after revision* 22 *October* 1975)

The author in the present paper derives two suitable formulae for determining the phase shifts for the well-known Yukawa potential function :

$$U(r) = \frac{-ge^{-\alpha r}}{r}.$$

where g (coupling constant) and α are real and positive. This potential function satisfies the first three necessary conditions as stipulated by Varshni (1957).

1. INTRODUCTION

In the last many years, there has been a great development in many directions in the general theory of scattering and the experimental studies in the nuclear and atomic collision. Many of these theoretical and experimental advances have been the result of their mutual stipulation. The problem of deducing a potential from the observed phase shifts has led to many mathematical investigations. However, it may be of interest, to obtain phase shifts from a given potential function, since, as it is pointed out by Schiff (1955, p. 106) that in the study of collisions, it is the phase shifts which completely determine the scattering and scattering cross-section, but to determine the phase shifts is a difficult task. With this motivation, the author, in the present paper, derives two suitable formulae for determining the phase shifts for the well-known Yukawa potential function :

$$U(r) = \frac{-ge^{-\alpha r}}{r} \qquad \text{...(1.1)}$$

where g (coupling constant) and α are real and positive. The potential function (1.1) satisfies the first three necessary conditions stipulated by Varshni (1957).

In our investigations, we require the following useful integral formulae due to Tietz [1963, p. 291 (14) and p. 292 (16)] :

$$\eta_l - \eta_{l+1} = \frac{\pi}{2k} \int_0^\infty r \frac{dU}{dr} J_{l+(1/2)}(kr)\, J_{l+(3/2)}(kr)\, dr \qquad \text{...(1.2)}$$

$$\eta_{l-1} - \eta_{l+1} = \frac{(l+\frac{1}{2})\pi}{k^2} \int_0^\infty \frac{dU}{dr} J^2_{l+(1/2)}(kr)\, dr \qquad \ldots(1.3)$$

where $J_\nu(kr)$ is the Bessel function of the first kind and the angular momentum quantum number l and the phase shifts difference $\eta_l - \eta_{l+1}$ are assumed to be small.

2. Derivations

The integrals (1.2) and (1.3) can further be simplified in view of the formula [Luke 1962, p. 24 (17)]:

$$J_\mu(az)\, J_\nu(bz) = \frac{\left(\frac{az}{2}\right)^\mu \left(\frac{bz}{2}\right)^\nu}{\Gamma(\mu+1)\,\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{az}{2}\right)^{2n}}{n!\,(\mu+1)_n}$$

$$\times\, {}_2F_1\left[\begin{matrix} -n, -\mu-n; \\ \nu+1; \end{matrix}\ \frac{b^2}{a^2}\right]. \qquad \ldots(2.1)$$

In (2.1), putting $a = b = k$, replacing z by r and summing the inner hypergeometric series ${}_2F_1$ with the help of Gauss formula (Rainville 1963) we obtain :

$$J_\mu(kr)\, J_\nu(kr) = \frac{\left(\frac{kr}{2}\right)^{\mu+\nu}}{\Gamma(\mu+1)\,\Gamma(\nu+1)}$$

$$\times\, {}_2F_3\left[\begin{matrix} \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2); \\ \mu+1, \nu+1, \mu+\nu+1; \end{matrix}\ -k^2r^2\right],$$

$$(\mathrm{Re}\,(\mu+\nu+1) > 0). \qquad \ldots(2.2)$$

In view of (2.2), (1.2) and (1.3) assumme the forms:

$$\eta_l - \eta_{l+1} = \frac{\pi k^{2l+1}}{2^{2l+3}\,\Gamma\left(l+\frac{3}{2}\right)\Gamma\left(l+\frac{5}{2}\right)} \int_0^\infty \frac{du}{dr}\, r^{2l+3}$$

$$\times\, {}_1F_2\left[\begin{matrix} l+2; \\ l+\frac{5}{2}, 2l+3; \end{matrix}\ -k^2r^2\right] dr,$$

$$(\mathrm{Re}\,(2l+3) > 0). \qquad \ldots(2.3)$$

$$\eta_{l-1} - \eta_{l+1} = \frac{\pi k^{2l-1}\left(l+\frac{1}{2}\right)}{2^{2l+1}\left[\Gamma\left(l+\frac{3}{2}\right)\right]^2} \int_0^\infty \frac{du}{dr} r^{2l+1}$$

$$\times\, {}_1F_2\left[\begin{matrix} l+1; \\ l+\frac{3}{2}, 2l+2; \end{matrix} \; -k^2r^2\right] dr,$$

$$(\text{Re}\,(2l+2) > 0). \qquad \text{...(2.4)}$$

Now substituting the value of du/dr in (2.3) and (2.4) from (1.1), expressing ${}_1F_2$ into the series form, interchanging the order of integration and summation permissible due to absolute convergence of the integrand, finally evaluating the inner gamma integrals and simplifying considerably, we arrive at our following main results :

$$\eta_l - \eta_{l+1} = \frac{\sqrt{\pi}\, g k^{2l+1}\,\Gamma(l+2)}{2\alpha^{2l+2}\,\Gamma\left(l+\frac{5}{2}\right)}\, {}_3F_2\left[\begin{matrix} l+2, l+2, l+\frac{3}{2}; \\ l+\frac{5}{2}, 2l+3; \end{matrix} \; -4\frac{k^2}{\alpha^2}\right]$$

$$+ \frac{\sqrt{\pi}\, g k^{2l+1}\,\Gamma(l+1)}{4\alpha^{2l+2}\,\Gamma\left(l+\frac{5}{2}\right)}$$

$$\times\, {}_3F_2\left[\begin{matrix} l+1, l+\frac{3}{2}, l+2; \\ l+\frac{5}{2}, 2l+3; \end{matrix} \; -4\frac{k^2}{\alpha^2}\right],$$

$$(\text{Re}\,(2l+3) > 0). \qquad \text{...(2.5)}$$

$$\eta_{l-1} - \eta_{l+1} = \frac{\sqrt{\pi}\, g k^{2l-1}\,\Gamma(l+1)}{2\alpha^{2l}\,\Gamma\left(l+\frac{3}{2}\right)}$$

$$\times\, {}_3F_2\left[\begin{matrix} l+1, l+1, l+\frac{1}{2}; \\ l+\frac{3}{2}, 2l+2; \end{matrix} \; -4\frac{k^2}{\alpha^2}\right]$$

(equation continued on p. 577)

$$+\frac{\sqrt{\pi}\, g k^{2l-1}\, \Gamma(l)}{4\alpha^{2l}\, \Gamma\left(l+\frac{3}{2}\right)} \times {}_3F_2\left[\begin{matrix} l,\ l+\frac{1}{2},\ l+1; \\ l+\frac{3}{2},\ 2l+2; \end{matrix}\ -4\frac{k^2}{\alpha^2}\right],$$

$$(\mathrm{Re}\,(2l+2) > 0). \qquad \ldots(2.6)$$

From eqns. (2.5) and (2.6), one may notice that for low energies and larger value of l, the phase shifts difference is an odd function of k. This is in agreement with an already known fact (Davydev 1965). Further, for physical interest, the phase shifts difference is to be finite, so that $\left|\frac{k^2}{\alpha^2}\right| < \frac{1}{4}$.

It can easily be seen that $\eta_0 - \eta_2$ obtained from (2.5) is exactly the same as obtained from (2.6). This can also be verified for other values of l. Thus formula (2.5) is equivalent to (2.6). Now if η_0 is known, (2.5) and (2.6) allow us to calculate approximate phase shifts for the Yukawa potential function.

Acknowledgement

The author is extremely grateful to Dr. R. C. Varma, for his kind supervision in the preparation of this paper. Thanks are also due to the refree for his valuable suggestions for the improvement of the paper.

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