THE SET COVERING PROBLEM WITH LINEAR FRACTIONAL FUNCTIONAL

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In this paper, techniques have been developed for finding an optimal cover solution of the set covering problem with linear fractional functional as its objective function. The techniques are enumerative and involve the evaluation of extreme points of a linear fractional functional programming problem in a given sequence. An example is given to illustrate the method.

INTRODUCTION

Recently, the set covering problem with linear objective function has been studied by Bellmore and Ratliff (1971). They developed a cutting plane method for solving such types of set covering problems. Lemke, Salkin and Spielberg (1971) developed another approach which is an enumerative one. Garfinkel and Nemhauser (1972) also studied the set covering problem.

In the present paper, the set covering problem with linear fractional functional as its objective function satisfying a condition is considered. The whole discussion is divided into four parts. Part I deals with the mathematical formulation of the problem, while the theoretical development is discussed in Part II. Algorithm I with an example is given in Part III. Part IV deals with the extreme point formulation of the problem with Algorithm II and the example.

PART I : MATHEMATICAL FORMULATION

Suppose an airline company has m flights to operate upon and n crews at its disposal, it being understood that a crew can handle at least one flight. Let $c_j > 0$ be the cost paid by the company when its jth crew is operated and let d_j be the commission that the company receives from the authorities when it employs its jth crew. Let $\beta > 0$ be the fixed amount paid to the company. Now, the company is interested in scheduling its crews in such a way that the cost is minimized and at the same time the profit is maximized, i.e., it is interested in determining a set of crews which would cover all the flights and for which $\frac{\Sigma\, c_j}{(\Sigma\, d_j + \beta)}$ is minimum.

Let $I = [1, 2, ..., m]$ be the set of m flights and $J = [1, 2, ..., n]$ be the set of n crews. Let P_j be the set of flights covered by the jth crew. Clearly, $P_j \subseteq I$. It is required to find a set of crews J^* which covers all the flights and minimizes the ratio

$$\frac{\sum_{j \in J^*} c_j}{\left(\sum_{j \in J^*} d_j + \beta\right)}$$

where $J^* \subseteq J$ and $\bigcup_{j \in J^*} P_j = I$.

Define a variable x_j associated with the jth crew as follows:

$$\begin{aligned} x_j &= 1 \text{ if } j\text{th crew is in the scheduling} \\ &= 0 \text{ otherwise} \end{aligned}$$

Define a_{ij} as follows:

$$\begin{aligned} a_{ij} &= 1 \text{ if } i\text{th flight is covered by the } j\text{th crew} \\ &= 0 \text{ otherwise.} \end{aligned}$$

Mathematically stated, the problem is:

Problem (I)

Minimize

$$Z = \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j + \beta}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geqslant 1 \qquad i = 1, 2, ..., m \qquad ...(1)$$

$$x_j = 0 \text{ or } 1 \qquad j = 1, 2, ..., n \qquad ...(2)$$

where $x_j = 0$ if j is not in the cover

$= 1$ if j is in the cover

$a_{ij} = 1$ if $i \in P_j$

$= 0$ otherwise.

This problem is the set covering problem with linear fractional functional as its objective function, which is to be studied in the present paper.

The linear fractional functional programming problem obtained by replacing (2) with $x_j \geqslant 0$ in problem (I) is:

Problem (II)

Minimize

$$Z = \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j + \beta}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geqslant 1 \quad i = 1, 2, \ldots, m$$

$$x_j \geqslant 0 \quad j = 1, 2, \ldots, n.$$

It is assumed that the solution set of problem (II) is regular and $(\sum_{j=1}^{n} d_j x_j + \beta)$ is strictly positive over this set.

Part II : Theoretical Development

Definitions

(i) *Cover solution* — A solution which satisfies (1) and (2) is called a cover solution of problem (I) and if it yields minimum Z, it is an optimal cover solution.

(ii) *Redundant column* — A column corresponding to j belonging to the cover J^* is said to be redundant if $J^* - \{j\}$ is also a cover and then the cover J^* is termed as redundant cover.

(iii) *Prime cover* — A cover J^* is said to be a prime cover, if none of the column corresponding to $j \in J^*$ is redundant.

Theorem I — A column corresponding to the suffix $\hat{j} \in J^*$ is not redundant, if there exists at least one $i \in P_{\hat{j}}$, such that

$$\sum_{j \in J^*} a_{ij} = 1.$$

Proof : If a column corresponding to $\hat{j}$ is redundant, then $J^* - \{\hat{j}\}$ will also be a cover, i.e., $\hat{j}$ can be removed from J^*. In such a case, all elements of $P_{\hat{j}}$ are members of some P_j's $j \in J^* - \{\hat{j}\}$. If $\hat{j}$ is not redundant, then there exists at least one member of $P_{\hat{j}}$, which is not present in any other $P_j \ \forall \ j \in J^* - \{\hat{j}\}$. Therefore, there exists at least one $i \in P_{\hat{j}}$ for which

$$a_{i\hat{j}} = 1$$

and

$$a_{ij} = 0 \;\forall\; j \in J^* - \{\hat{j}\}$$

which implies that for such an i,

$$\sum_{j \in J^*} a_{ij} = 1.$$

Hence, $\sum_{j \in J^*} a_{ij} = 1$ for at least one $i \in P_{\hat{j}}$.

Theorem II — Every optimal cover is a prime cover, if either (i) all d_j's are negative

or (ii) any ratio of the partial sums of c_j's and d_j's is greater than the value of the objective function at the cover solution and also the latter partial sum is positive

PROOF : Consider the optimal cover J'. If possible, assume that J' is not a prime cover. The value of the objective function for J' is

$$Z' = \frac{\sum_{j \in J'} c_j}{\left(\sum_{j \in J'} d_j + \beta\right)}.$$

Since J' is a redundant cover, a prime cover J'' can be derived from J' by removing the redundant columns of J'. The value of the objective function for J'' is

$$Z'' = \frac{\sum_{j \in J''} c_j}{\left(\sum_{j \in J''} d_j + \beta\right)}.$$

Case (i) — When all d_j's are negative.

In this case Z'' will be less than Z', which is justified as follows :

$$Z'' < Z' \text{ requires that } \frac{\sum_{j \in J''} c_j}{\sum_{j \in J''} d_j + \beta} < \frac{\sum_{j \in J'} c_j}{\sum_{j \in J'} d_j + \beta}$$

i.e., $$\left(\sum_{j \in J''} c_j\right)\left(\sum_{j \in J'} d_j + \beta\right) < \left(\sum_{j \in J'} c_j\right)\left(\sum_{j \in J''} d_j + \beta\right)$$

i.e., $$\left(\sum_{j \in J'} c_j - \sum_{j \in J'-J''} c_j\right)\left(\sum_{j \in J'} d_j + \beta\right)$$

$$< \left(\sum_{j \in J'} c_j\right)\left(\sum_{j \in J'} d_j - \sum_{j \in J'-J''} d_j + \beta\right)$$

i.e., $$\left(\sum_{j \in J'-J''} c_j\right)\left(\sum_{j \in J'} d_j + \beta\right) > \left(\sum_{j \in J'} c_j\right)\left(\sum_{j \in J'-J''} d_j\right)$$

which is always true because $(\sum_{j \in J'-J''} d_j)$ is negative and $(\sum_{j \in J'} d_j + \beta)$ is positive.

Case (*ii*) — When any ratio of the partial sum of c_j's and d_j's is greater than Z' and also the latter partial sum (i.e., partial sum of d_j's) is positive.

Again, as in case (i), $Z'' < Z'$ requires that

$$(\sum_{j \in J'-J''} c_j)(\sum_{j \in J'} d_j + \beta) > (\sum_{j \in J'} c_j)(\sum_{j \in J'-J''} d_j)$$

dividing it by positive number $(\sum_{j \in J'} d_j + \beta)(\sum_{j \in J'-J''} d_j)$ one gets

$$\frac{\sum_{j \in J'-J''} c_j}{\sum_{j \in J'-J''} d_j} > \frac{\sum_{j \in J'} c_j}{\sum_{j \in J'} d_j + \beta} = Z'$$

which is true because of the assumption of this case. Therefore, Z'' is always less than Z' under the conditions mentioned in cases (i) and (ii). Hence, J'' is a cover which is better than J', contradicting the fact that J' is an optimal cover. This shows that the assumption that J' is a redundant cover is wrong. Therefore, J' must be a prime cover under the condition mentioned in the above two cases.

Theorem III — If $J' = \{j/x_j = 1\}$ be any prime cover solution, then $X = \{x_j\}$ is an extreme point solution of problem (II).

PROOF : Since J' is a prime cover, none of the column corresponding to $j \in J'$ is redundant. If $J' = \{j^1, j^2, j^3, \ldots, j^k\}$, then by Theorem I, there exists at least one $i_1 \in P_{j^1}$, such that

$$a_{i_1 j^1} = 1$$

and

$$a_{i_1 j} = 0, \; j \in J' - \{j^1\}.$$

Therefore, the i_1th row in the matrix formed by columns corresponding to $j \in J'$ is of the form of a unit vector having unity at the j^1th place.

Similarly, for $i_2 \in P_{j^2}$, there exists an i_2th row which is also of the form of a unit vector having unity at j^2th place. Thus, the matrix fromed by k-columns specified by J' will contain k distinct unit vectors and hence there are k linearly independent columns. Hence, the columns specified by $j \in J'$ will form a basic feasible solution degenerate or non-degenerate, i.e., $X = \{x_j\}$, $j \in J'$ will be an extreme point of problem (II).

Inference — In Theorm II we have seen that the optimal cover is a prime cover and in Theorem III, it is established that the corresponding prime cover solution is an extreme point solution of problem (II). Based upon this, our algorithm investigates the extreme points of problem (II) in systematic order till a prime cover solution is reached, which will be the required optimal solution of problem (I).

PART III : ALGORITHM I

The first technique for solving the set covering problem consists of the following steps

Step I — Solve problem (II) by the method given by Martos (1960) and Swarup (1965a, b).

Let the set of its optimal basic feasible solutions be

$$X_1^{(1)} = [X_{11}^{(1)}, X_{12}^{(1)}, \dots X_{1p}^{(1)}].$$

If the variables in some member, say $X_{1p}^{(1)}$ of $X_1^{(1)}$, are zero-one, then an optimal solution is reached.

The optimal prime cover is

$$J' = \{j/x_{1p,j}^{(1)} = 1\}$$

where $x_{1p,j}^{(1)}$ is the jth component in $X_{1p}^{(1)}$. The corresponding value of the objective function is

$$\frac{\sum\limits_{j \in J'} c_j}{\sum\limits_{j \in J'} d_j + \beta} .$$

The optimal cover solution is $X_{1p}^{(1)}$.

If none of the members of $X_1^{(1)}$ is a zero-one solution, then proceed to step II.

Step II — Find the set of ith best extreme point solutions starting from $i = 2$ of problem (II). [For the method of finding the ith best extreme point refer to Murty (1968) and Puri (1974)]. If some member of $X_i^{(1)}$ is a zero-one solution, then terminate; otherwise go to step III.

Step III — Repeat step II for the next higher value of i, i.e., $i + 1$.

Convergence — The procedure will clearly converge in a finite number of steps, because it moves from one extreme point to another extreme point solution of problem (II), and these extreme points are always finite in number and none of them is repeated, because the value of the objective function is increased at every stage.

Example — Find an optimal prime cover solution for the following set covering a problem with linear fractional functional.

Minimize

$$Z = \frac{2x_1 + 3x_2 + 4x_3 + 5x_4}{-x_1 - 4x_2 - 3x_3 - 2x_4 + 12}$$

subject to

$$\begin{aligned} x_1 + x_2 &\geqslant 1 \\ x_2 + x_3 + x_4 &\geqslant 1 \\ x_1 + x_3 &\geqslant 1 \\ x_1, x_2, x_3, x_4 &= 0, 1. \end{aligned}$$

Solution — Here, $I = [1, 2, 3]$ $J = [1, 2, 3, 4]$

$$P_1 = [1, 3], P_2 = [1, 2], P_3 = [2, 3], P_4 = [2].$$

As all d_j's are negative and the denominator is strictly positive over the solution set of the linear fractional functional programming problem given below, case (i) is applicable.

The linear fractional functional programming problem is :

Minimize

$$Z = \frac{2x_1 + 3x_2 + 4x_3 + 5x_4}{-x_1 - 4x_2 - 3x_3 - 2x_4 + 12}$$

subject to

$$\begin{aligned} x_1 + x_2 &\geqslant 1 \\ x_2 + x_3 + x_4 &\geqslant 1 \\ x_1 + x_3 &\geqslant 1 \\ x_j &\geqslant 0 \quad j = 1, 2, 3, 4. \end{aligned}$$

Step I — Solve this problem. The optimal solution is

$$X_{11}^{(1)} = [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0]$$

As $X_{11}^{(1)}$ is not a zero-one solution, go to step II to find the 2nd best extreme point solution.

Step II — Set of the 2nd best extreme point solution is

$$X_2^{(1)} = [X_{21}^{(1)} = (1, 1, 0, 0)].$$

As $X_{21}^{(1)}$ is a zero-one solution, the process terminates and the prime cover is given by

$$J' = \{j/x_{21,j}^{(1)} = 1\} = [1, 2].$$

The value of the objective function is $Z = 5/7$.

The optimal cover solution is (1, 1, 0, 0).

Clearly, $P_1 \cup P_2 = I$.

Part IV : Extreme Point Formulation of the Problem

As $x_j = 0$ or 1 ($j = 1, 2, \ldots, n$) in the given problem (I), it follows that $X = \{x_j\}$ is an extreme point of a cube

$$I_n X \leqslant 1$$
$$X \geqslant 0.$$

Therefore, the equivalent extreme point formulation of the set covering problem is:

Minimize

$$Z = \frac{\sum_{j=1}^{n} c_j x_j}{\left(\sum_{j=1}^{n} d_j x_j + \beta\right)}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geqslant 1 \quad i = 1, 2, \ldots, m$$
$$x_j \geqslant 0 \quad j = 1, 2, \ldots, n$$

and X is an extreme point of

$$I_n X \leqslant 1$$
$$X \geqslant 0$$

i.e., $(x_1, x_2, \ldots, x_n)$ is an extreme point of

$$x_j \leqslant 1, \quad j = 1, 2, \ldots, n.$$

Clearly, every prime cover solution is an extreme point of

$$I_n X \leqslant 1$$
$$X \geqslant 0.$$

Consider the problem

Problem (III) $\begin{cases} \text{Minimize} \\ Z = \dfrac{\sum_{j=1}^{n} c_j x_j}{(\sum_{j=1}^{n} d_j x_j + \beta)} \\ \text{subject to} \\ I_n X \leqslant 1 \\ X \geqslant 0 \end{cases}$

Let

$$S = [X = (x_1, x_2, ..., x_n)/\sum_{j=1}^{n} a_{ij} x_j \geqslant 1, x_j \geqslant 0].$$

$$i = 1, 2, ..., m; \; j = 1, 2, ..., n.$$

Algorithm II

Step I — Solve the problem (III) and find the set $X_1^{(2)}$ of its optimal basic feasible solutions and go to step II.

Step II — If $X_1^{(2)} \cap S \neq \varphi$, then every $X \in X_1^{(2)} \cap S$ is an optimal prime cover solution. If $X_1^{(2)} \cap S = \varphi$, go to step III.

Step III — Find the set $X_2^{(2)}$ of the 2nd best extreme point solution of problem (III) by the method given in Murty (1968) and Puri (1974). If $X_2^{(2)} \cap S \neq \varphi$, every $X \in X_2^{(2)} \cap S$ is an optimal prime cover solution of problem (I). If $X_2^{(2)} \cap S = \varphi$ go to step IV.

Step IV — Find the set $X_i^{(2)}$ of *i*th best extreme point solutions starting from $i = 3$, till the feasibility of the given constraint set is satisfied, i.e., till $X_i^{(2)} \cap S \neq \varphi$.

Note — Investigation and derivation of extreme points of a cube is very simple and hence the approach works quite fast.

The same example when solved by this method is given below.

The extreme point formulation of the problem is:

Minimize

$$Z = \frac{2x_1 + 3x_2 + 4x_3 + 5x_4}{-x_1 - 4x_2 - 3x_3 - 2x_4 + 12}$$

subject to

$$\left.\begin{aligned} x_1 + x_2 &\geqslant 1 \\ x_2 + x_3 + x_4 &\geqslant 1 \\ x_1 + x_3 &\geqslant 1 \\ x_1,\ x_2,\ x_3,\ x_4 &\geqslant 0 \end{aligned}\right\} \qquad \ldots (1)$$

and (x_1, x_2, x_3, x_4) is an extreme point of

$$\begin{aligned} x_1 &\leqslant 1 \\ x_2 &\leqslant 1 \\ x_3 &\leqslant 1 \\ x_4 &\leqslant 1 \\ x_1, x_2, x_3, x_4 &\geqslant 0. \end{aligned}$$

Consider the following problem :

Minimize

$$Z = \frac{2x_1 + 3x_2 + 4x_3 + 5x_4}{-x_1 - 4x_2 - 3x_3 - 2x_4 + 12}$$

subject to

$$\begin{aligned} x_1 &\leqslant 1 \\ x_2 &\leqslant 1 \\ x_3 &\leqslant 1 \\ x_4 &\leqslant 1 \\ x_1, x_2, x_3, x_4 &\geqslant 0. \end{aligned}$$

On solving the problem, we have its optimal solution

$$X_1^{(2)} = [X_{11}^{(2)} = (0, 0, 0, 0)].$$

Now, $S = [X = (x_1, x_2, \ldots, x_n)/(X \text{ satisfies } (1)]$.

Clearly, $X_1^{(2)} \cap S = \varphi$; therefore, find $X_2^{(2)}, X_3^{(2)}, \ldots$. We have $X_i^{(2)} \cap S = \varphi$, $i = 2, 3, 4, 5$ and

$$X_6^{(2)} = [X_{6,1}^{(2)} = (1, 1, 0, 0)].$$

Now, $X_6^{(2)} \cap S \neq \varphi$.

$$X_6^{(2)} \cap S \neq \varphi. = (1, 1, 0,0). = X_{6,1}^{(2)}.$$

Hence, the process terminates and the optimal cover solution is $X_{6,1}^{(2)} = (1, 1, 0, 0)$ and the optimal prime cover is $J^1 = [1, 2]$.

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