SOME OPERATIONAL PROPERTIES OF GENERALIZED LEGENDRE TRANSFORM AND THEIR APPLICATIONS II

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The present communication is aimed to extend the applicability of the generalized Legendre transform to a new class of boundary value problems. The range of generalized Legendre transform is converted into 0 to 1 instead of -1 to 1, by defining even and odd generalized Legendre transforms. In the process of developing these transforms a new integral involving product of spheroidal wave angle function of order zero has been evaluated for both n even and odd. In the end, the applicability of the transforms developed in this paper is exhibited by solving the problem of heat transfer in a slab bounded between parallel planes with variable conductivity and heat generation within it.

The well known even and odd Legendre transforms due to Tranter follow as the particular cases of the transforms developed herein.

Laplace, Fourier, Hankel, Mellin and Legendre transforms have been applied to the solution of boundary value problems of Mathematical Physics. The application of such transforms often reduces a partial differential equation in n independent variables to one in $n - 1$ variables and it is often possible by successive operation of this type, to reduce the problem to the solution of an ordinary differential equation. Most of the work done has involved problems in which the range of variable has been $(0, \infty)$, but Doetsch (1935) has extended the theory to finite ranges in the case of Fourier transforms, Sneddon (1946) has discussed finite Hankel transforms and more recently Tranter (1950) and Churchill (1954) have developed finite Legendre transforms. Sneddon has also suggested extensions to transforms with other kernels, but he gives no example.

Recently the authors (Gupta and Gupta 1976) have developed a new transform with the spheroidal wave angle function, $S_{mn}(c, \eta)$ as kernel under the name 'generalized Legendre transforms'. The aim of this paper is to discuss some operational properties of this transform so that this transform may be applied to a wider class of boundary value problems, and thus we have developed even generalized Legendre transform and odd generalized Legendre transform. The advantage of use of the generalized Legendre transforms is that they reduce to the analysis involved in the solution of boundary value problems almost to a drill and bring the work into line with the recently published work of this type involving other integral transforms, e.g., Laplace, Fourier, Legendre transforms.

In the end, the odd and even generalized Legendre transforms have been applied to solve the problem of heat transfer in a slab bounded between parallel planes. This type of mathematical approach may be useful in deciding upon the depth at which the pipe lines must be buried so that whatever may be conditions on the ground level, oil or liquid flowing in the pipe will not freeze. This has application in very severely cold regions.

1. INTEGRAL

Consider the integral,

$$I = \int_0^1 S_{0n}(c, \eta)\, S_{0n'},(c, \eta)\, d\eta$$

where $S_{0n}(c, \eta)$ is the spheroidal wave angle function of the first kind of the order zero and satisfies the differential equation

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta} S_{0n}(c, \eta)\right\} + [\lambda_{0n}(c) - c^2\eta^2]\, S_{0n}(c, \eta). \qquad \ldots(1.1)$$

$S_{0n}(c, \eta)$ can be expanded in a series of Legendre functions $P^0_n(\eta)$, given by equation,

$$S_{0n}(c, \eta) = \sum_{r=0,1}^{\infty}{}' d_r^{0n}(c)\, P_r(\eta).$$

Now

$$I = \int_0^1 \sum_{r=0,1}^{\infty}{}' d_r^{0n}(c)\, P_r(\eta) \sum_{s=0,1}^{\infty}{}' d_s^{0n'}(c)\, P_s(\eta)\, d\eta$$

$$= \sum_{r=0,1}^{\infty}{}' d_r^{0n}(c) \sum_{s=0,1}^{\infty}{}' d_s^{0n'}(c) \int_0^1 P_r(\eta)\, P_s(\eta)\, d\eta.$$

Here term by term integration is justified because of the uniform convergence of the spheroidal wave angle function.

Hence

$$\int_0^1 S_{0n}(c, \eta)\, S_{0n'}(c, \eta)\, d\eta = 0 \qquad \ldots(1.2a)$$

when $r - s$ is even i.e. when $n - n'$ is even and

$$\int_0^1 S_{0n}(c,\eta)\, S_{0n'}(c,\eta)\, d\eta = \sum_{r=0,1}^{\infty}{}' \left[d_r^{0n}(c)\right]^2 \int_0^1 \left[P_r(\eta)\right]^2 d\eta \quad \text{...(1.2b)}$$

when $n = n'$ (Whittaker and Watson 1965, p. 306). Hence

$$\int_0^1 S_{0n}(c,\eta)\, S_{0n'}(c,\eta)\, d\eta$$

$$= \sum_{r=0}^{\infty} \left[d_{2r}^{0,2n}(c)\right]^2 \frac{1}{4r+1} = \tfrac{1}{2} N_{0,2n} \quad \text{...(1.3)}$$

when n is even

$$= \sum_{r=0}^{\infty} \left[d_{2r+1}^{0,2n+1}(c)\right]^2 \frac{1}{4r+3} = \tfrac{1}{2} N_{0,2n+1} \quad \text{...(1.4)}$$

when n is odd [Flammer 1957, p. 22, eqn. (3.1.33)].

2. Representation and Inversion Theorems of Even Generalized Legendre Transform and Odd Generalized Legendre Transform

Here we use the integral considered above to define even generalized Legendre transform and odd generalized Legendre transform as:

Theorem 1 — If $F(x)$ is a continuous, single valued function defined over the interval $0 \leqslant x \leqslant 1$, then we define the even generalized Legendre transform of $F(x)$ by

$$\bar{f}_{0,2n}(c) = \int_0^1 F(x)\, S_{0,2n}(c,x)\, dx. \quad \text{...(2.1)}$$

By generalized Fourier series expansion we have

$$F(x) = \sum_{n=0}^{\infty} A_n S_{0,2n}(c,x)$$

where

$$A_n = \frac{2}{N_{0,2n}} \int_0^1 F(x)\, S_{0,2n}(c,x)\, dx = 2\bar{f}_{0,2n}(c)/N_{0,2n}.$$

Hence the inversion formula for this transform is given by

$$F(x) = \sum_{n=0}^{\infty} \frac{2\bar{f}_{0,2n}(c)\, S_{02n}(c, x)}{N_{0,2n}} \qquad \text{...(2.2)}$$

where $N_{0,2n}$ is given by equation,

$$N_{mn} = 2 \sum_{r=0,1}^{\infty}{}' \frac{(r + 2m)!}{(2r + 2m + 1)!\,(r)!} \left[d_r^{mn}(c)\right]^2 . \qquad \text{...(2.3)}$$

[Flammer 1957, p. 22, eqn. (3.1.33)]

when $m = 0$ and $n = 2n$.

Theorem 2 — If $F(x)$ satisfies the conditions stated for the validity of Theorem 1, then we define odd generalized Legendre transform of $F(x)$ by

$$\bar{f}_{0,2n+1}(c) = \int_0^1 F(x)\, S_{0,2n+1}(c, x)\, dx \qquad \text{...(2.4)}$$

and hence by the help of generalized Fourier series expansion and eqns. (1.2) and (1.4) the inversion formula for this transform is obtained to be,

$$F(x) = 2 \sum_{n=0}^{\infty} \frac{\bar{f}_{0,2n+1}(c)\, S_{0,2n+1}(c, x)}{N_{0,2n+1}} \qquad \text{...(2.5)}$$

where $N_{0,2n+1}$ is the normalization factor for $S_{0,2n+1}(c, x)$ and is given by eqn. (2.3) when $m = 0$ and n is replaced by $2n + 1$.

For $c = 0$ these transforms are in complete agreement with the even and odd Legendre transforms given by Tranter (1950) and thus the transforms due to Tranter are the particular cases of our transforms.

3. Basic Operational Properties of the Transforms Defined in Theorems 1 and 2

Equation (1.1) can be written in the self adjoint form as,

$$[L_x + \lambda_{0n}(c)]\, S_{0n}(c, x) = 0 \qquad \text{...(3.1)}$$

where

$$L_x = \frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial}{\partial x}\right\} - c^2x^2. \qquad \text{...(3.2)}$$

Now even generalized Legendre transform of $[L_x F(x)]$ is given by

$$T_e\left\{L_x F(x)\right\} = \int_0^1 \left[\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial F}{\partial x}\right\} - c^2x^2F\right] S_{0,2n}(c, x)\, dx$$

$$= \left[(1-x^2)\frac{\partial F}{\partial x} S_{0,2n}(c, x) - F(x)(1-x^2)\frac{\partial}{\partial x} S_{0,2n}(c, x)\right]_0^1$$

$$+ \int_0^1 F\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial}{\partial x} S_{0,2n}(c, x)\right\} dx$$

(on integrating twice by parts)

$$= \left[-S_{0,2n}(c, x)\frac{\partial F}{\partial x}\right]_{x=0} - \lambda_{0,2n}(c)\bar{f}_{0,2n}(c)$$

$$= \left[\left(\frac{\partial F}{\partial x}\right)_{x=0}\frac{(-1)^{n+1}(2n)!}{2^{2n}(n!)^2}\right] - \lambda_{e,2n}(c)\bar{f}_{0,2n}(c).$$

Similarly the odd transform of $[L_x F(x)]$ is given by

$$T_0\left[L_x F(x)\right] = \left[F(x)\frac{\partial}{\partial x} S_{0,2n+1}(c, x)\right]_{x=0} - \lambda_{0,2n+1}(c)\bar{f}_{0,2n+1}(c)$$

$$= \left[F(0)\frac{(-1)^n(2n+2)!}{2^{2n+1}n!(n+1)!}\right] - \lambda_{0,2n+1}(c)\bar{f}_{0,2n+1}(c).$$

We have thus the following:

Theorem 3 — If $F(x)$ and $F'(x)$ be bounded in the range $0 \leqslant x \leqslant 1$, $F''(x)$ be bounded and integrable in each of the subinterval $0 < x < 1$ then even and odd generalized Legendre transforms of $F(x)$ exist and if,

$$\lim_{x\to 1}(1-x^2)F(x) = \lim_{x\to 1}(1-x^2)F'(x) = 0$$

then even and odd transforms of $[L_x F(x)]$ exist and are given by

$$T_e\left[L_x F(x)\right] = \left[+\left(\frac{\partial F}{\partial x}\right)_{x=0}\frac{(-1)^{n+1}(2n)!}{2^{2n}(n!)^2}\right] - \lambda_{0,2n}(c)\bar{f}_{0,2n}(c) \quad \ldots(3.3)$$

and

$$T_0\left\{L_x F(x)\right\} = \left[F(0)\frac{(-1)^n(2n+2)!}{2^{2n+1}n!(n+1)!}\right] - \lambda_{0,2n+1}(c)\bar{f}_{0,2n+1}(c). \quad \ldots(3.4)$$

Hence the Generalized Legendre transforms may be applied to obtain the solution of the boundary value problems, involving the differential form

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial}{\partial x}\right\} - c^2x^2$$

for the range $0 \leqslant x \leqslant 1$.

4. An Application of Odd Generalized Legendre Transforms

To discuss the application of odd generalized Legendre transforms we consider the unsteady state of flow of heat inside a slab bounded by two parallel planes between $x = 0$ and $x = a$.

Statement of the Problem

We consider the unsteady state of diffusion of heat inside a slab of variable conductivity bounded by the planes $x = 0$ and $x = a$, when there is a source of heat generation within the slab. The variable conductivity K being given by

$$K = K_1(a^2 - x^2)$$

and the source of heat generation, Q being given by

$$Q = K_1\{\varphi_1(x, t) - c^2x^2\, u(x, t)\} \qquad \ldots(4.1)$$

where φ_1 is a function of x and t, being given by

$$\varphi_1(x, t) = \varphi(t)\,\psi(x)$$

and $u(x, t)$ is the temperature function.

We would like to remark that when the expression (2.1) gives negative value it becomes sink in place of source of heat generation.

The surface $x = a$ of the slab is insulated, while the surface $x = 0$ is maintained at a constant temperature. Initially the temperature is supposed to be a prescribed function of x.

Hence we get the equations

$$K_1\frac{\partial}{\partial x}\left[(a^2 - x^2)\frac{\partial u}{\partial x}\right] - K_1c^2x^2u = -K_1\varphi(t)\,\psi(x) + \rho c_1\frac{\partial u}{\partial t}$$

$$\left.\begin{array}{lll} \text{(i)}\quad K_1(a^2 - x^2)\dfrac{\partial u}{\partial x} = 0 & \text{at } x = a & t > 0 \\ \text{(ii)}\quad u(x, t) = u_0K_1 & \text{at } x = 0 & t > 0 \\ \text{(iii)}\quad u(x, t) = F'(x) & 0 \leqslant x \leqslant a & t = 0. \end{array}\right\}$$

Making use of non-dimensional quantities i.e. putting

$$u = K_1u'u_0;\; x = ax';\; t = \frac{\rho c_1 t'}{K_1};\; c^2a^2 = (c')^2$$

and dropping primes we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} - c^2x^2u = -\varphi(t)\,\psi(x) + \frac{\partial u}{\partial t} \qquad \ldots(4.2)$$

where

$$\varphi(t) = \frac{1}{u_0K_1}\,\varphi'\left(\frac{\rho c_1 t'}{K_1}\right)$$

and

$$\psi(x) = \psi'(x'a)$$

with boundary conditions,

$$\left.\begin{array}{llll} \text{(i)} & (1-x^2)\dfrac{\partial u}{\partial x} = 0 & x = 1 & t \geqslant 0 \\ \text{(ii)} & u(x, t) = 1 & x = 0 & t \geqslant 0 \\ \text{(iii)} & U(x, t) = F(x) & 0 < x < 1 & t = 0 \end{array}\right\} \qquad \ldots(4.3)$$

where

$$F(x) = F'(ax').$$

Applying the odd generalized Legendre transform defined by eqn. (2.4) to eqns. (4.2) and (4.3) we get

$$\frac{d\bar{u}}{dt} + \lambda_{0,2^n+1}(c)\,\bar{u} = \frac{(-1)^n(2n+2)!}{2^{2n+1}n!\,(n+1)!} + \bar{\psi}\varphi(t) \qquad \ldots(4.4)$$

where

$$\left.\begin{array}{l} \bar{u} = \displaystyle\int_0^1 u(x, t)\,S_{0,2^n+1}(c, x)\,dx \\ \bar{\psi} = \displaystyle\int_0^1 \psi(x)\,S_{0,2^n+1}(c, x)\,dx \end{array}\right\} \qquad \ldots(4.5)$$

and

$$\bar{u} = \bar{f}_{0,2^n+1}(c) \quad \text{when } t = 0. \qquad \ldots(4.6)$$

Hence

$$\bar{u} = Ae^{-\lambda_{0,2n+1}(c)t} + \frac{(-1)^n(2n+2)!}{\lambda_{0,2^n+1}(c)\,n!\,(n+1)!\,2^{2n+1}}$$

$$+\,\bar{\psi}e^{-\lambda_{0,2n+1}(c)t}\int_0^t e^{\lambda_{0,2n+1}(c)\tau}\,\varphi(\tau)\,d\tau$$

and to determine the constant A we have the condition (4.6).

Hence

$$\bar{u} = e^{-\lambda_{0,2n+1}(c)t} \int_0^1 F(x)\, S_{0,2n+1}(c, x)\, dx$$

$$+ \frac{(-1)^n (2n+2)!}{2^{2n+1}\, n!\, (n+1)!} \left[1 - e^{-\lambda_{0,2n+1}(c)t}\right]$$

$$+ \bar{\psi} e^{-\lambda_{0,2n+1}(c)t} \int_0^t e^{\lambda_{0,2n+1}(c)\tau} \varphi(\tau)\, d\tau. \qquad \ldots(4.7)$$

Hence using inversion formula (2.5) we get

$$u(x, t) = \sum_{n=0}^{\infty} \frac{2\bar{u}\, S_{0,2n+1}(c, x)}{N_{0,2n+1}} \qquad \ldots(4.8)$$

where $\bar{u}$ is given by equation (4.7).

Particular Forms of $\psi(x)$ *and* $\varphi(t)$

(A) Suppose $\psi(x) = x^{\lambda-1}$, $\lambda \geqslant 1$ and

$\varphi(t) = \sin t.$

Then

$$\bar{\psi} = \sum_{r=0}^{\infty} d_{2r+1}^{0,2n+1}(c) \int_0^1 x^{\lambda-1} P_{2r+1}(x)\, dx$$

$$= \sum_{r=0}^{\infty} d_{2r+1}^{0,2n+1}(c) \frac{\sqrt{\pi}\, 2^{-\lambda}\, \Gamma\lambda}{\Gamma\left(\frac{\lambda}{2} - r\right) \Gamma\left(\frac{\lambda}{2} + r + \frac{3}{2}\right)}$$

[Erdélyi *et al.* (1954), p. 314, eqn. (5)]

and

$$\int_0^t e^{\lambda_{0,2n+1}(c)\tau}\, \varphi(\tau)\, d\tau$$

$$= \frac{e^{\lambda_{0,2n+1}(c)t}}{1 + \lambda_{0,2n+1}^2(c)} \left[1 - \cos t + \lambda_{0,2n+1}(c) \sin t\right].$$

Hence

$$\bar{u} = e^{-\lambda_{0,2n+1}(c)t}\, \bar{f}_{0,2^n+1}(c) + \frac{1}{\lambda_{0,2^n+1}(c)} \frac{(-1)^n (2n+2)!}{2^{2n+1}\, n!\, (n+1)!} \times [1 - e^{-\lambda_{0,2n+1}(c)t}]$$

$$+ \frac{[1 - \cos t + \lambda_{0,2^n+1}(c) \sin t]}{[1 + \lambda^2_{0,2^n+1}(c)]} \sum_{r=0}^{\infty} d^{0,2^n+1}_{2r+1}(c) \frac{\sqrt{\pi}\, 2^{-\lambda}\Gamma\lambda}{\Gamma\left(\frac{\lambda}{2}-r\right)\Gamma\left(\frac{\lambda}{2}+r+\frac{3}{2}\right)}$$

where

$\bar{f}_{0,2^n+1}(c)$ is the odd generalized Legendre transform of $F(x)$, i.e.

$$\bar{f}_{0,2^n+1}(c) = \int_0^1 F(x)\, S_{0,2^n+1}(c, x)\, dx.$$

Hence using the inversion formula (2.5) we get

$$u(x, t) = \sum_{n=0}^{\infty} 2\Big[e^{-\lambda_{0,2n+1}(c)t} + \frac{(-1)^n (2n+1)!}{\lambda_{0,2^n+1}(c)\, 2^{2n+1}\, n!\, (n+1)!} \times \{1 - e^{-\lambda_{0,2n+1}(c)t}\}$$

$$+ \left\{\frac{1 - \cos t + \lambda_{0,2^n+1}(c) \sin t}{1 + \lambda^2_{0,2^n+1}(c)\, t}\right\}$$

$$\times \sum_{r=0}^{\infty} d^{0,2^n+1}_{2r+1}(c) \frac{\sqrt{\pi}\, \Gamma\lambda}{2^{\lambda}\, \Gamma\left(\frac{\lambda}{2}-r\right)\Gamma\left(\frac{\lambda}{2}+\frac{3}{2}+r\right)}\Big] \frac{S_{0,2^n+1}(c,x)}{N_{0,2^n+1}} \qquad \ldots(4.9)$$

In case $\lambda = 1$ and $F(x) = 0$, then

$$u(x, t) = \sum_{n=0}^{\infty} 2\Big[\frac{(-1)^n}{\lambda_{0,2^n+1}(c)} \left\{1 - \exp(-\lambda_{0,2^n+1}(c)\, t)\right\} \frac{(2n+2)!}{2^{2n+1} n!\, (n+1)!}$$

$$+ \frac{d_1^{0,2n+1}(c)}{2} \left\{\frac{1 - \cos t + \lambda_{0,2^n+1}(c) \sin t}{1 + \lambda^2_{0,2^n+1}(c)}\right\}\Big] \frac{S_{0,2^n+1}(c, x)}{N_{0,2^n+1}}. \qquad \ldots(4.10)$$

(B) If $\psi(x) = x^{\lambda-1}(1-x^2)^{\mu}$ $\lambda > 1$ $\mu > 0$ and

$\varphi(t) = \sin t$

then

$$\bar{\psi} = \sum_{r=0}^{\infty} d_{2r+1}^{0,2^n+1}(c) \int_0^1 x^{\lambda-1}(1-x^2)^{\mu} P_{2r+1}(x)\, dx$$

$$= \sum_{r=0}^{\infty} d_{2r+1}^{0,2^n+1}(c) \frac{\Gamma(1+\mu)\,\Gamma\left(\frac{\lambda}{2}\right)}{2\Gamma\left(1+\mu+\frac{\lambda}{2}\right)}$$

$$\times\, {}_3F_2 \left[\begin{matrix} r+1,\ -r-\tfrac{1}{2},\ 1+\mu; \\ 1,\ 1+\mu+\dfrac{\lambda}{2}; \end{matrix} \; 1 \right]$$

[Erdélyi *et al.* 1954, p. 314, eqn. (6)]

and as in case (A)

$$\int_0^t \exp\left(\lambda_{0,2^n+1}(c)\, \tau\right) \varphi(\tau)\, d\tau$$

$$= \exp\left(\lambda_{0,2^n+1}(c)\, t\right) \frac{[1-\cos t + \lambda_{0,2^n+1}(c) \sin t]}{[1+\lambda_{0,2^n+1}^2(c)]}.$$

Hence

$$u(x,t) = \sum_{n=0}^{\infty} 2\Bigg[\exp\left\{\lambda_{0,2^n+1}(c)\, t\right\} \bar{f}_{0,2^n+1}(c)$$

$$+ \frac{(-1)^n (2n+2)!}{\lambda_{0,2^n+1}(c)\, 2^{2n+1}\, n!\, (n+1)!} \left\{1 - \exp\left(-\lambda_{0,2^n+1}(c)\, t\right)\right\}$$

$$+ \left\{ \frac{1-\cos t + \lambda_{0,2^n+1}(c)\sin t}{1+\lambda_{0,2^n+1}^2(c)} \right\}$$

$$\times \sum_{r=0}^{\infty} d_{2r+1}^{0,2^n+1}(c) \frac{\Gamma(1+\mu)\,\Gamma\left(\frac{\lambda}{2}\right)}{2\Gamma\left(1+\mu+\frac{\lambda}{2}\right)}$$

(*equation continued on p.* 599)

$$\times\ {}_3F_2\left\{\begin{matrix} r+1,\ -r-\frac{1}{2},\ 1+\mu; \\ 1,\ 1+\mu+\frac{\lambda}{2}; \end{matrix}\ 1\right\}\Bigg]\frac{S_{0,2n+1}(c, x)}{N_{0,2n+1}} \qquad \ldots(4.11)$$

(C) If $F(x) = 0$; $\psi(x) = 1$; $\varphi(t) = 1$, then

$$\bar{\psi} = \int_0^1 S_{0,2n+1}(c, x)\, dx$$

$$= \sum_{r=0}^{\infty} d_{2r+1}^{0,2n+1}(c) \int_0^1 P_{2r+1}(x)\, dx$$

$$= 0$$

and

$$\int_0^t \exp\left(\lambda_{0,2n+1}(c)\, \tau\right) \varphi(\tau)\, d\tau = \frac{\exp\left(\lambda_{0,2n+1}(c)\, t\right) - 1}{\lambda_{0,2n+1}(c)}.$$

Hence in this case the temperature function $u(x, t)$ is given by

$$u(x, t) = \sum_{n=0}^{\infty}\left[\frac{(-1)^n (2n+2)!}{2^{2n}\, n!\, (n+1)!\, \lambda_{0,2n+1}(c)}\left\{1 - \exp\left(-\lambda_{0,2n+1}(c)\, t\right)\right\}\right] \times \frac{S_{0,2n+1}(c, x)}{N_{0,2n+1}}. \qquad \ldots(4.12)$$

5. An Application of Even Generalized Legendre Transform

As an application of the even transform, we consider the same problem, as in the preceding article, when the surface $x = 1$ is insulated as before and the flux over the surface $x = 0$ is constant.

Hence in this case we have to solve eqn. (4.2) under the boundary conditions

$$\left.\begin{array}{lll} \text{(i)}\ (1 - x^2)\dfrac{\partial u}{\partial x} = 0, & x = 1; & t > 0 \\ \text{(ii)}\ \dfrac{\partial u}{\partial x} = 1, & x = 0; & t > 0 \\ \text{(iii)}\ u(x, t) = F(x), & 0 \leqslant x \leqslant 1; & t = 0. \end{array}\right\} \qquad \ldots(5.1)$$

Applying the even generalized Legendre transform, defined by eqn. (2.1) to eqns. (4.2) and (5.1) and using the operational property given by eqn. (3.3), we get,

$$\frac{d\bar{u}}{dt} + \lambda_{0,2n}(c)\,\bar{u} = \frac{(-1)^{n+1}(2n)!}{2^{2n}(n!)^2} + \bar{\psi}_e \varphi(t) \qquad \ldots(5.2)$$

where $\bar{u}$ and $\bar{\psi}_e$ are the even generalized Legendre transforms of $u(x, t)$ and $\psi(x)$ respectively.

Hence proceeding as in section 4, we get

$$\bar{u} = \exp(\lambda_{0,2n}(c)\,t)\,\bar{f}_{2n}(c) - \frac{(-1)^n(2n)!}{\lambda_{0,2n}(c)\,2^{2n}(n!)^2}\left\{1 - \exp(-\lambda_{0,2n}(c)\,t\right\}$$

$$+ \bar{\psi}_e \exp(-\lambda_{0,2n}(c)\,t) \int_0^t \exp(\lambda_{0,2n}(c)\,\tau)\,\varphi(\tau)\,d\tau. \qquad \ldots(5.3)$$

Using inversion formula (2.2), we get

$$u(x, t) = \sum_{n=0}^{\infty} \frac{2\bar{u}\,S_{0,2n}(c, x)}{N_{0,2n}} \qquad \ldots(5.4)$$

where $\bar{u}$ is given by (5.3).

Particular Case

If $F(x) = 0$; $\psi(x) = 1$; $\varphi(t) = 1$, then

$$\bar{\psi}_e = \int_0^1 S_{0,2n}(c, x)\,dx = d_0^{0,2n}(c)$$

and

$$\int_0^t \exp(\lambda_{0,2n}(c)\,\tau)\,\varphi(\tau)\,d\tau = \frac{\exp(\lambda_{0,2n}(c)\,t) - 1}{\lambda_{0,2n}(c)}.$$

Hence in this case the temperature function $u(x, t)$ is given by

$$u(x, t) = \sum_{n=0}^{\infty} 2\left[\frac{(-1)^{n+1}(2n)!}{\lambda_{0,2n}(c)\,2^{2n}(n!)^2} + \frac{d_0^{0,2n}(c)}{\lambda_{0,2n}(c)}\right]$$

$$\times\left[1 - e^{-\lambda_{0,2n}(c)t}\right]\frac{S_{0,2n}(c, x)}{N_{0,2n}}. \qquad \ldots(5.5)$$

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