ON THE ABSOLUTE NÖRLUND SUMMABILITY FACTORS OF INFINITE SERIES

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In this paper a theorem on absolute Nörlund summability factors has been established. It extends the earlier results of Pati (1962), Lal (1963) and Siya Ram (1971) on absolute Cesàro, absolute harmonic and absolute Nörlund summability factors of infinite series, respectively.

§ 1. Let $\{S_n\}$ be the sequence of the partial sums of an infinite series $\Sigma\, a_n$, and $\{p_n\}$ be a sequence of constants such that

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n, \quad P_{-1} = p_{-1} = 0.$$

The transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} S_\nu = \frac{1}{P_n} \sum_{\nu=0}^{n} P_{n-\nu} a_\nu, \quad (P_n \neq 0),$$

defines the sequence $\{t_n\}$ of the Nörlund means of the sequence $\{S_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\Sigma\, a_n$ is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation (Mears 1935), that is to say

$$\sum_n |t_n - t_{n-1}| < \infty.$$

Two important cases of Nörlund summability are (Hardy 1949, § 5.13):

(i) *Harmonic summability*: When

$$p_n = \frac{1}{n+1}, \text{ so that } P_n \sim \log n, \text{ as } n \to \infty,$$

and

(ii) *Cesàro summability*: When

$$p_n = \binom{n+\delta-1}{\delta-1}, \quad \delta > 0.$$

If

$$\sum_{\nu=1}^{n} \frac{|S_\nu|}{\nu} = O(\log n),$$

as $n \to \infty$, then Σa_n is said to be strongly bounded by logarithmic means, or bounded $[R, \log n, 1]$.

§ 2. Lal (1963) established the following theorem:

Theorem A—If

$$\sum_{\nu=1}^{n} \frac{|S_\nu|}{\nu} = O(\log n),$$

as $n \to \infty$, then the series $\Sigma a_n \lambda_n \log(n+1)/n$, where $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n/n$ is convergent, is absolutely summable $(N, 1/(n+1))$.

Recently, Siya Ram (1971) generalized the above theorem of Lal by establishing the following theorem:

Theorem B—Let $p_0 > 0$ and $\{p_n\}$ be a non-negative and non-increasing sequence. If

$$\sum_{\nu=1}^{n} \frac{|S_\nu|}{\nu} = O(\chi_n),$$

as $n \to \infty$, when χ_n is a positive non-decreasing sequence and if ϵ_n is such that

$$\sum_{\nu=1}^{n} \nu \,|\Delta^2 \epsilon_\nu|\, \chi_\nu = O(1)$$

and

$$|\epsilon_n|\, \chi_n = O(1)$$

then the series $\Sigma a_n \epsilon_n P_n/(n+1)$ is summable $|N, p_n|$.

§ 3. The object of this paper is to effect further generalization of the above theorem of Siya Ram by establishing the following theorem:

Theorem—Let $p_0 > 0$ and $\{p_n\}$ be a non-negative and non-increasing sequence. If

$$\sum_{\nu=1}^{n} \lambda_\nu \,|S_\nu| = O(\chi_n)$$

as $n \to \infty$, where $\{\chi_n\}$ is a positive non-decreasing sequence and $\{\lambda_n\}$ is a positive decreasing sequence such that

$$n \triangle \lambda_n = O(\lambda_n) \qquad \ldots(3.1)$$

and if the sequence $\{\epsilon_n\}$ is such that

$$\sum_{\nu=1}^{n} \nu \mid \triangle^2 \epsilon \nu \mid X_\nu = O(1), \qquad \ldots(3.2)$$

and

$$\mid \epsilon_n \mid X_n = O(1), \qquad \ldots(3.3)$$

as $n \to \infty$, then the series $\Sigma\, a_n P_n \epsilon_n \lambda_n$ is summable $\mid N, p_n \mid$.

It is interesting to note that, if we put in our theorem $\lambda_n = 1/n$; $p_n = 1/n$, so that $P_n \sim \log n$, as $n \to \infty$, $X_n = \log n$ and $\{\epsilon_n\}$ a convex sequence such that $\Sigma\, \epsilon_n/n < \infty$, then the theorem of Lal becomes a special case of our theorem. Putting $\lambda_n = 1/n$ and using conditions (3.2) and (3.3) our theorem becomes the theorem of Siya Ram.

If we put $\lambda_n = 1/n$, $X_n = \log n$, $p_n = 1$ and $\{\epsilon_n\}$ a convex sequence such that $\Sigma\, \epsilon_n/n < \infty$, in our theorem, the following theorem due to Pati (1962) becomes a special case of our theorem:

Theorem C—Let $\{\epsilon_n\}$ be a convex sequence such that $\Sigma\, \epsilon_n/n$ is convergent. If $\Sigma\, a_n$ is bounded $[R, \log n, 1]$, then $\Sigma\, a_n \epsilon_n$ is summable $\mid C, 1 \mid$.

§ 4. To prove the theorem we shall require the following lemmas:

Lemma 1 (Ahmad 1966)—For $p_0 > 0$ and $\{p_n\}$ a non-negative and monotonic non-increasing sequence and for $\nu \geqslant 1$

$$\sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}} \leqslant \frac{K}{\nu} \qquad \ldots(4.1)$$

$$\sum_{n=\nu}^{\infty} \frac{p_n (P_n - P_{n-\nu})}{P_n P_{n-\nu}} \leqslant K \qquad \ldots(4.2)$$

$$\sum_{n=\nu}^{\infty} \frac{\mid \triangle_n p_{n-\nu-1} \mid}{P_{n-1}} \leqslant \frac{K}{P_\nu} + \frac{K}{\nu} \qquad \ldots(4.3)$$

and

$$\sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}} \leqslant K \qquad \ldots(4.4)$$

where K is some constant.

Lemma 2 (Ahmad 1973)—If $\{X_n\}$ be a positive monotonic non-decreasing sequence, and if $\{\epsilon_n\}$ is a sequence such that $|\epsilon_n| \chi_n = O(1)$, $n\triangle\chi_n = O(\chi_n)$, then $\Sigma n\chi_n |\triangle^2\epsilon_n| < \infty$, implies that

$$\Sigma |\triangle\epsilon_n| \chi_n < \infty \qquad \text{...(4.5)}$$

and

$$m |\triangle\epsilon_m| \chi_m = O(1), \quad \text{as} \quad m \to \infty. \qquad \text{...(4.6)}$$

§ 5. *Proof of Theorem*: Without any loss of generality we may assume that $a_0 = 0$. Let T_n denote the nth Nörlund mean of the series $\Sigma a_n P_n \epsilon_n \lambda_n$. Then by definition

$$T_n - T_{n-1} = \frac{1}{P_n} \sum_{\nu=0}^{n} P_{n-\nu} u_\nu - \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} P_{n-1-\nu} u_\nu,$$

where $u_\nu = a_\nu P_\nu \epsilon_\nu \lambda_\nu$

$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} [P_\nu p_{n-\nu} - P_{n-\nu} p_n] u_\nu$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) u_\nu + \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} (p_{n-\nu} - p_n) u_\nu$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \triangle_\nu [(P_n - P_{n-\nu}) P_\nu \epsilon_\nu \lambda_\nu] S_\nu$$

$$+ \frac{p_n}{P_n P_{n-1}} (P_n - P_0) P_n \epsilon_n \lambda_n S_n$$

$$+ \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} \triangle_\nu [(p_{n-\nu} - p_n) P_\nu \epsilon_\nu \lambda_\nu] S_\nu$$

$$+ \frac{1}{P_{n-1}} (p_0 - p_n) P_n \epsilon_n \lambda_n S_n.$$

Therefore

$$|T_n - T_{n-1}| \leqslant \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} \triangle_\nu [P_n - P_{n-\nu}) P_\nu |\epsilon_\nu| \lambda_\nu] |S_\nu|$$

$$+ \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} \triangle_\nu [(p_{n-\nu} - p_n) P_\nu |\epsilon_\nu| \lambda_\nu] |S_\nu|$$

$$+ p_0 |\epsilon_n| . \lambda_n |S_n|$$

$$= M_1(n) + M_2(n) + M_3(n), \text{ say.} \qquad \text{...(5.1)}$$

Now

$$\sum_{n=1}^{\infty} M_1(n) = O(1) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \Big[\sum_{\nu=1}^{n} (P_n - P_{n-\nu}) p_{\nu+1} \mid \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid$$

$$+ \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) P_\nu \mid \triangle \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid$$

$$+ \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) P_\nu \mid \epsilon_\nu \mid . \triangle \lambda_\nu \mid S_\nu \mid$$

$$+ \sum_{\nu=1}^{n} p_{n-\nu} P_\nu \mid \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid \Big]$$

$$= O(1) \sum_{\nu=1}^{\infty} p_{\nu+1} \mid \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu})$$

$$+ O(1) \sum_{\nu=1}^{\infty} P_\nu \mid \triangle \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu})$$

$$+ O(1) \sum_{\nu=1}^{\infty} P_\nu \mid \epsilon_\nu \mid . \triangle \lambda_\nu \mid S_\nu \mid \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu})$$

$$+ O(1) \sum_{\nu=1}^{\infty} P_\nu \mid \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid \sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}}$$

$$= O(1) \sum_{\nu=1}^{\infty} p_{\nu+1} \mid \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid + O(1) \sum_{\nu=1}^{\infty} P_\nu \mid \triangle \epsilon_\nu \mid \lambda_\nu \mid S_\nu \mid$$

$$+ O(1) \sum_{\nu=1}^{\infty} P_\nu \mid \epsilon_\nu \mid . \triangle \lambda_\nu \mid S_\nu \mid$$

$$+ O(1) \sum_{\nu=1}^{\infty} \frac{P_\nu}{\nu} \mid \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid,$$

by (4.1) and (4.2)

$$= O(1) \sum_{\nu=1}^{\infty} \mid \epsilon_\nu \mid . \lambda_\nu \mid S_\nu \mid + O(1) \sum_{\nu=1}^{\infty} \mid \triangle \epsilon_\nu \mid \lambda_\nu \mid S_\nu \mid . \nu$$

$$+ O(1) \sum_{\nu=1}^{\infty} \mid \epsilon_\nu \mid . \triangle \lambda_\nu \mid S_\nu \mid . \nu$$

as $P_n/n = O(1)$ and $\{p_n\}$ is bounded

$$= L_1 + L_2 + L_3, \text{ say.} \qquad \ldots(5.2)$$

Using (3.3) and (4.5), we have

$$\sum_{\nu=1}^{n} |\epsilon_\nu| \, |\lambda_\nu S_\nu| = O(1) \sum_{\nu=1}^{n-1} |\triangle \epsilon_\nu| \chi_\nu + O(1) |\epsilon_n| \chi_n + O(1)$$

$$= O(1), \text{ as } n \to \infty.$$

Thus

$$L_1 = O(1). \qquad \ldots(5.3)$$

Again, using (3.2), (4.5) and (4.6), we have

$$\sum_{\nu=1}^{n} |\triangle \epsilon_\nu| \,.\, \nu \,.\, |\lambda_\nu S_\nu|$$

$$= O(1) \sum_{\nu=1}^{n-1} \triangle \{ |\triangle \epsilon_\nu| \,.\, \nu \} \chi_\nu + O(1) |\triangle \epsilon_n| n \chi_n + O(1)$$

$$= O(1) \sum_{\nu=1}^{n-1} |\triangle^2 \epsilon_\nu| \nu \chi_\nu + O(1) \sum_{\nu=1}^{n-1} |\triangle \epsilon_\nu| \chi_\nu + O(1)$$

$$= O(1), \text{ as } n \to \infty,$$

so that

$$L_2 = O(1). \qquad \ldots(5.4)$$

Lastly, using (3.1), we have

$$\sum_{\nu=1}^{n} |\epsilon_\nu| \, |\triangle \lambda_\nu| \, |S_\nu| \nu = \sum_{\nu=1}^{n} \left\{ \frac{|\triangle \lambda_\nu| \, |\epsilon_\nu|}{\lambda_\nu} \,.\, \nu \right\} |\lambda_\nu S_\nu|$$

$$= O\left(\sum_{\nu=1}^{n} |\epsilon_\nu| \, |\lambda_\nu S_\nu| \right),$$

which is the same as L_1.

Thus,

$$L_3 = O(1). \qquad \ldots(5.5)$$

Again

$$\sum_{n=1}^{\infty} M_2(n) = O(1) \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \Big[\sum_{\nu=1}^{n} (p_{n-\nu} - p_n) p_{\nu+1} |\epsilon_\nu| \lambda_\nu |S_\nu|$$

$$+ \sum_{\nu=1}^{n} (p_{n-\nu} - p_\nu) P_\nu |\triangle \epsilon_\nu| \lambda_\nu |S_\nu| +$$

[*equation continued on page* 66]

$$+ \sum_{\nu=1}^{n} (p_{n-\nu} - p_\nu) P_\nu |\epsilon_\nu| . \triangle \lambda_\nu| |S_\nu|$$

$$+ \sum_{\nu=1}^{n} (p_{n-\nu} - p_{n-\nu-1}) P_\nu |\epsilon_\nu| . \lambda_\nu |S_\nu| \Big]$$

$$= O(1) \sum_{\nu=1}^{\infty} p_{\nu+1} |\epsilon_\nu| . \lambda_\nu |S_\nu| \sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}}$$

$$+ O(1) \sum_{\nu=1}^{\infty} P_\nu | \triangle \epsilon_\nu | . \lambda_\nu |S_\nu| \sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}}$$

$$+ O(1) \sum_{\nu=1}^{\infty} P_\nu |\epsilon_\nu| . \triangle \lambda_\nu |S_\nu| \sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}}$$

$$+ O(1) \sum_{\nu=1}^{\infty} P_\nu |\epsilon_\nu| . \lambda_\nu |S_\nu| \sum_{u=\nu}^{\infty} \frac{\triangle_n (p_{n-\nu-1})}{P_{n-1}}$$

$$= O(1) \sum_{\nu=1}^{\infty} |\epsilon_\nu| \, |\lambda_\nu S_\nu| + O(1) \sum_{\nu=1}^{\infty} | \triangle \epsilon_\nu | . \nu . | \lambda_\nu S_\nu |$$

$$+ O(1) \sum_{\nu=1}^{\infty} |\epsilon_\nu| . \nu \triangle \lambda_\nu | S_\nu |,$$

using (4.3) and (4.4)

$= L_1 + L_2 + L_3$, from (5.2) and noting that $P_n/n = O(1)$, as $n \to \infty$, and L_1, L_2 and L_3 have already been treated, so that

$$\sum_{n=1}^{\infty} M_2(n) = O(1).$$

Finally

$$\sum_{n=1}^{\infty} M_3(n) = O(1) \sum_{n=1}^{\infty} p_0 |\epsilon_n| \, |\lambda_n S_n|$$

$= O(1)$, as it has already been treated.

Thus we get the desired result.

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