OPERATIONAL CALCULUS OF SPHEROIDAL WAVE ANGLE FUNCTION (GENERALIZED LEGENDRE TRANSFORM)

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The spheroidal wave angle function has been used as kernel to develop a new finite integral transform and the corresponding inversion series. Some operational properties of the transform have been discussed and the transform of some particular function have been obtained and are given in a tabular form. The well-known Legendre Transform follows as the particular case of the transform aimed at.

Tranter (1950) and Churchill (1954) introduced Legendre transform of $F(x)$ over the range $(-1, 1)$ with Legendre polynomial of first kind, $P_n(x)$, as kernel and extended its utility to a wide class of partial differential equations. The aim of the investigation carried out in this paper is to generalize the results of Churchill (1954) to spheroidal wave angle functions of the first kind, $S_{mn}(c, \eta)$, over the same range. It seems proper to use this transform under the name 'generalized Legendre transform'. Generalized Legendre transforms of some particular functions have been calculated and are compiled in tabular form.

In the end, some of its applications to various problems by way of solving the partial differential equations have been discussed also. The results obtained here are quite new. This problem may be an approximate mathematical model of a combustion chamber in diesel/petrol engines.

1. Spheroidal Wave Angle Functions of the First Kind

For ready reference we give here the definition and some of the properties of spheroidal wave angle function of the first kind, $S_{mn}(c, \eta)$ (as given by Flammer 1957).

The function $S_{mn}(c, \eta)$ is the solution of the second order differential equation,

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dS_{mn}}{d\eta}\right] + \left[\lambda_{mn}(c) - c^2\eta^2 - \frac{m^2}{1-\eta^2}\right] S_{mn}(c, \eta) = 0 \qquad \ldots(1.1)$$

and can be expressed as

$$S_{mn}(c, \eta) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) P_{m+r}^{m}(\eta), \qquad \text{...(1.2)}$$

$$m = 0, 1, 2 \ldots, \; n = 0, 1, 2 \ldots$$

where $P_n^m(\eta)$ is the associated Legendre function of the first kind. The coefficients $d_r^{mn}(c)$ are given by Flammer [1957, eqns. (3.1.31*a*) and (3.1.31*b*), p. 21].

For $m = 0$ and $c = 0$ the function reduces to

$$S_{0n}(0, \eta) = P_n(\eta). \qquad \text{...(1.3)}$$

The function is orthogonal over the interval $(-1, 1)$ and the normalization factor, N_{mn}, is given by equation

$$N_{mn} = 2 \sum_{r=0,1}^{\infty}{}' \frac{(r + 2m)!}{(2r + 2m + 1)!\,(r)!} \left[d_r^{mn}(c)\right]^2 \qquad \text{...(1.4)}$$

[Flammer 1957, eqns. (3.1.33), p. 22]

The equation (1.1) can be written in self-adjoint form as

$$[L_\eta + \lambda_{mn}(c)]\, S_{mn}(c, \eta) = 0 \qquad \text{...(1.5}a\text{)}$$

where the self-adjoint linear operator L_η is given by

$$L_\eta = \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\frac{\partial}{\partial\eta}\right] - \frac{m^2}{1 - \eta^2} - c^2\eta^2. \qquad \text{...(1.5}b\text{)}$$

when $c = 0$; $\lambda_{mn}(c) = n(n + 1)$ for $n \geqslant m$.

2. Representation Theorem and Inversion Formula for Generalized Legendre Transform.

Theorem 1 — If any function $F(x)$ is continuous and single valued over the range $(-1, 1)$ then its generalized Legendre transform is defined by

$$T\{F(x)\} = \bar{f}_{mn}(c) = \int_{-1}^{1} F(x)\, S_{mn}(c, x)\, dx. \qquad \text{...(2.1)}$$

For $m = 0$ and $c = 0$, this transform reduces to the Legendre transform of Churchill (1954) and thus the Legendre transform is a particular case of this transform.

By generalized Fourier series $F(x)$ can be expanded in the form

$$F(x) = \sum_{n=0}^{\infty} A_n S_{mn}(c, x) \qquad \text{...(2.2)}$$

where

$$A_n = \frac{1}{N_{mn}} \int_{-1}^{1} F(x)\, S_{mn}(c, x)\, dx$$

$$= \frac{1}{N_{mn}} \bar{f}_{mn}(c) \qquad \ldots(2.3)$$

where N_{mn} is the normalization factor of $S_{mn}(c, \eta)$ and is given by eqn. (1.4).

Hence the inversion formula is given by

$$F(x) = \sum_{n=0}^{\infty} \frac{\bar{f}_{mn}(c)\, S_{mn}(c, x)}{N_{mn}}. \qquad \ldots(2.4)$$

Theorem 2 — If $F(x)$ and $F'(x)$ be bounded in the interval $-1 \leqslant x \leqslant 1$; $F''(x)$ be bounded and integrable in each of the subinterval of $-1 < x < 1$, then generalized Legendre transform of $F(x)$ exists and if

$$\lim_{x \to \pm 1} (1 - x^2)\, F(x) = \lim_{x \to \pm 1} (1 - x^2)\, F'(x) = 0 \qquad \ldots(2.5)$$

then generalized Legendre transform of $L_x [F(x)]$ exists and is given by

$$T\{L_x F(x)\} = -\lambda_{mn}(c)\, \bar{f}_{mn}(c) \qquad \ldots(2.5a)$$

for every integral values of m and n and for every value of c.

PROOF : $T\{L_x F(x)\}$

$$= \int_{-1}^{1} \left[\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial F}{\partial x}\right\} - \frac{m^2 F}{1 - x^2} - c^2 x^2 F\right] S_{mn}(c, x)\, dx$$

$$= \int_{-1}^{1} \left[\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial F}{\partial x}\right\} S_{mn}(c, x)\right] dx$$

$$- \int_{-1}^{1} \left[\frac{m^2 F}{1 - x^2} + c^2 x^2 F\right] S_{mn}(c, x)\, dx.$$

The first integral is integrated twice by parts to give

$$\int_{-1}^{1} \left[\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial F}{\partial x}\right\}\right] S_{mn}(c, x)\, dx =$$

(*equation continued on p.* 605)

$$= \left[(1-x^2)\frac{\partial F}{\partial x} S_{mn}(c,x) - (1-x^2)\frac{\partial}{\partial x} S_{mn}(c,x)\,F\right]_{-1}^{1}$$

$$+ \int_{-1}^{1} \left[\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial}{\partial x} S_{mn}(c,x)\right\}\right] F(x)\,dx.$$

The first term vanishes on both the limits because of the condition (2.5), hence

$$T\{L_x F(x)\} = \int_{-1}^{1} \left[\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial}{\partial x} S_{mn}(c,x)\right\}\right] F(x)\,dx$$

$$- \int_{-1}^{1} \left[\frac{m^2 S_{mn}(c,x)}{1-x^2} + c^2x^2 S_{mn}(c,x)\right] F(x)\,dx$$

$$= \int_{-1}^{1} F(x)\left[\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial}{\partial x}\right\} - \frac{m^2}{1-x^2} - c^2x^2\right] S_{mn}(c,x)\,dx$$

$$= - \int_{-1}^{1} F(x)\,\lambda_{mn}(c)\,S_{mn}(c,x)\,dx$$

[By virtue of equation (1.5*a*)]

$$= -\lambda_{mn}(c)\,\bar{f}_{mn}(c).$$

Hence the theorem.

Formula (2.5*a*) represents the basic operational property of the generalized Legendre transform, defined above under which the differential operator $[L_x F(x)]$, defined by equation (1.5*b*), is transformed into an algebraic operator $-\lambda_{mn}(c)\,\bar{f}_{mn}(c)$.

Hence the transform defined in the foregoing article can be applied to get the solution of boundary value problems.

3. Some Operational Properties of Generalized Legendre Transform

(a) If $T\{F(x)\}$ and $T\{G(x)\}$ exists, then

$$T\{C_1F(x) + C_2G(x)\} = C_1T\{F(x)\} + C_2T\{G(x)\}$$

where C_1 and C_2 are constants.

This property is obvious because of the linearity of the transform defined herein.

(b) If each of the functions $F(x)$ and $L_xF(x)$ satisfies the conditions stated for the validity of Theorem 1 then transform of the iterated differential operator $L_x[L_xF(x)]$ can be written as

$$
\begin{aligned}
T\{L_x^2 F(x)\} &= T[L_x\{L_x F(x)\}] \\
&= -\lambda_{mn}(c)\, T\{L_x F(x)\} \\
&= -\lambda_{mn}(c)\, \{-\lambda_{mn}(c)\, \bar{f}_{mn}(c)\} \\
&= (-1)^2\, [\lambda_{mn}(c)]^2\, \bar{f}_{mn}(c)
\end{aligned}
$$

and more generally,

$$T\{L_x^p F(x)\} = (-1)^p\, [\lambda_{mn}(c)]^p\, \bar{f}_{mn}(c).$$

(c) $T\{F(x) + A\} = \bar{f}_{mn}(c) + 0$ when $m = 0, 1$ and r is odd

$$
= \bar{f}_{mn}(c) + A
$$

$$
\times \sum_{r=0}^{\infty} \frac{d_{2r}^{mn}(c)\, x 2^m\, \Gamma\left(1 + \frac{m}{2}\right) \Gamma\left(1 - \frac{m}{2}\right)}{\Gamma\left(1 - \frac{r}{2}\right) \Gamma\left(\frac{1}{2} - \frac{r}{2} - m\right) \Gamma\left(\frac{4 + m + r}{2}\right) \Gamma\left(1 - \frac{m + r}{2}\right)}
$$

when $m = 0, 1$ and r is even [Erdélyi 1954, p. 316, eqn. (6)].

For higher values of m the calculations do not seem to be straightforward.

4. Transform of Some Particular Functions

(i) Let $F(x) = S_{mn'}(c, x)$.

Then by the orthogonal property, (1.4), of $S_{mn}(c, x)$ the generalized Legendre transform of $F(x)$ is given by

$$\bar{f}_{mn}(c) = \frac{1}{N_{mn}}\, \delta_{nn'}. \qquad \ldots(4.1)$$

(ii) Let $F(x) = Q_k^m(x)$, $k \geqslant m$

where $Q_k^m(x)$ is the associated Legendre function of the second kind. Then

$$\bar{f}_{mn}(c) = \int_{-1}^{1} Q_k^m(x)\, S_{mn}(c, x)\, dx$$

(*equation continued on p.* 607)

$$= \int_{-1}^{1} \sideset{}{'}\sum_{r=0,1}^{\infty} d_r^{mn}(c)\, P_{m+r}^{m}(x)\, Q_k^m(x)\, dx$$

$$= \sideset{}{'}\sum_{r=0,1}^{\infty} d_r^{mn}(c) \int_{-1}^{1} P_{m+r}^{m}(x)\, Q_k^m(x)\, dx$$

[Term by term integration is valid because of the uniform convergence of series involved]

$$= \sideset{}{'}\sum_{r=0,1}^{\infty} d_r^{mn}(c) \frac{(-1)^m [1-(-1)^{k+m+r}(k+m)!]}{(m+r-k)(m+r+k+1)(k-m)!}$$

[Erdélyi 1954, eqn. (29), p. 279].

...(4.2)

(iii) Let $F(x) = P_n^m(x),\ m \leqslant n.$

Then

$$\bar{f}_{mn}(c) = \sideset{}{'}\sum_{r=0,1}^{\infty} d_r^{mn}(c) \int_{-1}^{1} P_n^m(x)\, P_{m+r}^m(x)\, dx$$

$$= d_{n-m}^{mn}(c) \int_{-1}^{1} [P_n^m(x)]^2\, dx$$

$$= d_{n-m}^{mn}(c) \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \qquad \text{...(4.3)}$$

[Erdélyi 1954, eqns. (27) and (28), p. 279].

(iv) Let $F(x) = x^k(z-x)^{-1}(1-x^2)^{m/2},\ |z| > 1.$

Then

$$\bar{f}_{mn}(c) = \sideset{}{'}\sum_{r=0,1}^{\infty} d_r^{mn}(c) \int_{-1}^{1} x^k(z-x)^{-1}(1-x^2)^{m/2} P_{m+r}^m(x)\, dx$$

$$= (-2)^m z^k (z^2-1)^{m/2} \sideset{}{'}\sum_{r=0,1}^{\infty} d_r^{mn}(c)\, Q_{m+r}^m(z) \qquad \text{...(4.4)}$$

[Erdélyi 1954, eqn. (26), p. 279].

(v) Let $F(x) = (z - x)^{-1} (1 - x^2)^{-m/2}$, $m \geqslant 0$, $|z| > 1$.

Then

$$\bar{f}_{mn}(c) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) \int_{-1}^{1} (z - x)^{-1} (1 - x^2)^{-m/2} P_{m+r}^m (x)\, dx$$

$$= 2e^{-im\pi}(z^2 - 1)^{-m/2} \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c)\, Q_{m+r}^m(z) \qquad \ldots(4.5)$$

[Erdélyi 1954, eqn. (17), p. 316].

(vi) Let $F(x) = (1 - x^2)^{m/2} P_j^m(x) P_k^m(x)$, $2m \leqslant k - j$, $j \geqslant m$, $k \geqslant m$.

Then

$$\bar{f}_{mn}(c) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) \int_{-1}^{1} (1 - x^2)^{m/2} P_j^m(x) P_k^m(x) P_{m+r}^m(x)\, dx$$

$$= \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) \frac{(-1)^m (k+m)!\,(j+m)!\,(r+2m)!\,(s-m)!}{\pi^{3/2}(k-m)!\,(j-m)!\,r!\,(s-k)!}$$

$$\times \frac{\Gamma(t - k + \frac{1}{2})\,\Gamma(m + \frac{1}{2})\,\Gamma(t - j + \frac{1}{2})\,\Gamma(t - m - r + \frac{1}{2})}{(s-j)!\,(s-m-r)!\,\Gamma(s + \frac{3}{2})} \qquad \ldots(4.6)$$

where $2s = k + j + 2m + r$, $2t = k + j + r$ [Erdélyi 1954, eqn. (32), p. 280].

(vii) Let $F(x) = (1 - x^2)^{(\mu/2) - m - (1/2)} \left(\frac{1-x}{1+x}\right)^{z/2} P_\lambda^m(x)$.

Then

$$\bar{F}_{mn}(c) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) \int_{-1}^{1} (1 - x^2)^{(\mu/2) - m - (1/2)} \left(\frac{1-x}{1+x}\right)^{z/2}$$

$$\times P_\lambda^m(x) P_{m+r}^m(x)\, dx$$

$$= \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c)(-1)^m 2^{2m+\mu-1} B\left(\frac{\mu}{2} + \frac{z}{2} + \frac{1}{2}; \frac{\mu}{2} - \frac{z}{2} + \frac{1}{2}\right)$$

$$\times \sum_{s=0}^{[\frac{1}{2}(\lambda+r-m)]} C_s(2\lambda - 2m + 2r - 4s + 1)\, F_{\lambda-m+r-2s}^{\mu}(z) \qquad \ldots(4.7)$$

where

$$C_s = \sum_{j=0}^{\min[\lambda-m,\, r]} \frac{(\lambda+m+r-j)\,(m+\tfrac{1}{2})_j\,(\tfrac{1}{2})_\lambda\,(\tfrac{1}{2})_{m+r-j}}{(\tfrac{1}{2})_{\lambda+r-j+1}\,(\lambda-m-j)!\,(r-j)!\,(\lambda+m+r-2j)!\,(j)!}$$

$$\times \frac{(\lambda+r-m-2j)\,(m)_{s-j}\,(-\tfrac{1}{2}-\lambda-2m-r+2j)_{m+s-j}}{(s-j)!\,(\tfrac{1}{2}-\lambda-r+2j)_{s-j}\,(\lambda-m-j)!}$$

[Shabde 1940, result (3.4)].

All these results have been compiled in tabular form (Table I).

TABLE I

Table of generalized Legendre transforms

$F(x)$	$\int_{-1}^{1} F(x)\, S_{mn}(c, x)\, dx$
(1) $S_{mn'}(c, x)$	$\frac{1}{N_{mn}}\,\delta_{nn'}$
(2) $Q_k^m(x)$, $k \geqslant m$	$\sum_{r=0,1}^{\infty}{}' \; d_r^{mn}(c)\,(-1)^m \frac{1-(-1)^{m+k+r}\,(k+m)!}{(m+r-k)(m+r+k+1)\,(k-m)!}$
(3) $P_n^m(x)$, $m \leqslant n$	$\frac{2d_{n-m}^{mn}(c)\,(n+m)!}{(2n+1)\,(n-m)!}$
(4) $x^k(z-x)^{-1}$ $\times (1-x^2)^{m/2}$, $\|z\| > 1$, $k=0, 1, 2\ldots$	$(-2)^m z^k (z^2-1)^{m/2} \sum_{r=0,1}^{\infty}{}' \; d_r^{mn}(c)\, Q_{m+r}^m(z)$
(5) $(z-x)^{-1}$ $\times (1-x^2)^{-m/2}$, $m \geqslant 0$, $\|z\| > 1$	$2e^{-im\pi}(z^2-1)^{-m/2} \sum_{r=0,1}^{\infty}{}' \; d_r^{mn}(c)\, Q_{m+r}^m(z)$

(*continued on p.* 610)

TABLE I (*continued*)

(6) $(1-x^2)^{m/2} P_j^m(x) \times P_k^m(x)$, $j \geqslant m;\ k \geqslant m;$ $2m \leqslant k-j$	$\dfrac{(-1)^m (k+m)!\,(j+m)!\,\Gamma(m+\frac{1}{2})}{\pi^{3/2}(k-m)!\,(j-m)!} \times \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) \left[\dfrac{(r+2m)!\,(s-m)!\,\Gamma(t-k+\frac{1}{2})\,\Gamma(t-j+\frac{1}{2}) \times \Gamma(t-m-r+\frac{1}{2})}{(r)!\,(s-k)!\,(s-j)!\,(s-m-r)!\,\Gamma(s+\frac{3}{2})}\right]$ (where $2s = k+j+2m+r;\ 2t = k+j+r$)
(7) $(1-x^2)^{(\mu/2)-m-\frac{1}{2}} \times \left(\dfrac{1-x}{1+x}\right)^{z/2} P_\lambda^m(x)$	$(-1)^m 2^{2m+\mu-1} B\left(\dfrac{\mu}{2}+\dfrac{z}{2}+\frac{1}{2};\dfrac{\mu}{2}-\dfrac{z}{2}+\frac{1}{2}\right) \times \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) \sum_{s=0}^{(\lambda+r+m)/2} C_s(2\lambda+2m+2r-4s+1) F^{\mu}_{\lambda+r-m-2s}(z)$ where $C_s = \sum_{j=0}^{\min(\lambda-m,r)} \dfrac{(\lambda+m+r-j)!\,(m+\frac{1}{2})_j(\frac{1}{2})_{\lambda-j}(\frac{1}{2})_{m+r-j}}{(\frac{1}{2})_{\lambda+r-j+1}(\lambda-m-j)!\,(r-j)!\,(\lambda+m+r-2j)!} \times \dfrac{(\lambda+r-m-2j)!\,(m)_{s-j}(-\frac{1}{2}-\lambda-2m-r+2j)_{m+s-j}}{(j)!\,(s-j)!\,(\frac{1}{2}-\lambda-r+2j)_{s-j}}$

REFERENCES

Churchill, R. V. (1954). The Operational Calculus of Legendre Transforms. *J. math. Phys.*, **33**, 165-78.

Erdélyi, A. *et al.* (1953). Tables of Integral Transforms, Vol. I. McGraw-Hill Book Co., Inc., New York.

——————— (1954). Tables of Integral Transforms, Vol. II. McGraw-Hill Book Co., Inc., New York.

Flammer, C. (1957). Spheroidal Wave Functions. Stanford University Press, Stanford.

Shabde, N. G. (1940). On some results involving Legendre functions. *Bull. Calcutta math. Soc.*, **32**, 121-24.

Sneddon, I. N. (1951). Fourier Transforms. McGraw-Hill Book Co., Inc., New York.

Tranter, C. J. (1950). Legendre transform. *Q. Jl Math. Oxford*, **121**, 1-8.